

Buckling transition and boundary layer in non-Euclidean plates

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Non-Euclidean plates are thin elastic bodies having no stress-free configuration, hence exhibiting residual stresses in the absence of external constraints. These bodies are endowed with a three-dimensional reference metric, which may not necessarily be immersible in physical space. Here, based on a recently developed theory for such bodies, we characterize the transition from flat to buckled equilibrium configurations at a critical value of the plate thickness. Depending on the reference metric, the buckling transition may be either continuous or discontinuous. In the infinitely thin plate limit, under the assumption that a limiting configuration exists, we show that the limit is a configuration that minimizes the bending content, among all configurations with zero stretching content (isometric immersions of the midsurface). For small but finite plate thickness, we show the formation of a boundary layer, whose size scales with the square root of the plate thickness and whose shape is determined by a balance between stretching and bending energies.

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I. INTRODUCTION

The classical literature on thin elastic bodies deals primarily with two types of bodies—*plates* and *shells*. Mathematically, a plate can be viewed as a continuous stack of identical flat surfaces glued together, whereas a shell can be viewed as a continuous stack of nonidentical (and not necessarily flat) surfaces glued together. The term non-Euclidean plate was coined in [1] to describe thin elastic bodies which—like plates—do not exhibit structural variations across their thin dimension and yet—unlike plates—do not have a planar rest configuration. Such elastic bodies can neither be described as shells, which bear structural variations across their thin dimension (e.g., shells do not display reflectional symmetry about the midsurface) and possess curved stress-free rest configurations. Non-Euclidean plates exhibit residual stresses even in the absence of external constraints and are therefore inherently frustrated.

Elastic bodies having such properties are ubiquitous in biology. Growing tissues, such as plant leaves, are relatively thin elastic structures that may exhibit complex equilibrium configurations in the absence of external forces [2,3]. In addition, thin elastic bodies which have no stress-free configuration have been engineered in a laboratory [4], for example, the environmentally responsive gels shown in Fig. 1.

There are various ways to treat elastic bodies which exhibit residual stress. One way is to treat the residual stress as a physical field and characterize its properties [5]. Another is to decompose the deformation gradient into a product of a plastic (or growth) process, which deforms the body from a rest configuration into some “virtual” configuration, and an elastic relaxation from the virtual configuration to the current configuration [6,7]. A third approach is to decompose the strain tensor additively into a plastic (or growth) strain, leading to a virtual configuration, and an elastic strain [8,9]. The treatment presented here for residually stressed three-dimensional bodies is very similar to the third approach. All deformations which are not of elastic nature are completely

ignored, i.e., it is assumed that the virtual configuration which the growth or plastic deformation led to is known, and the appropriate elastic relaxation is solved. This in turn enables us to treat large “plastic strains” in a noniterative manner.

A static theory of non-Euclidean plates was developed in [1] following the fundamental principles laid by Truesdell [10] and its modern interpretation by Ciarlet and Gratie [11] and Ciarlet [12]. The starting point in [1] is the formulation of a covariant three-dimensional elasticity theory in the form of an energy functional. A first notable property of this energy functional is its expression in terms of the three-dimensional metric of the configuration. Specifically, the energy density is quadratic in the deviation of the metric from a reference metric. This deviation of the metric is a strain, which reduces to the standard Green-Saint Venant strain for

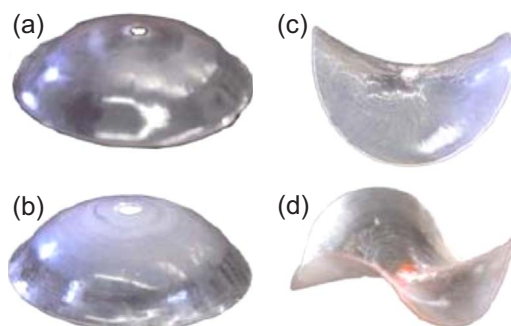


FIG. 1. (Color online) Four elastic plates made of thermoresponsive gel as described in [4]. All four structures bear no structural variation across their thickness. Their radius is 3 cm. The midsurface of the [(a) and (b)] positively curved disks possess a reference metric of constant Gaussian curvature $K=0.11 \text{ cm}^{-2}$. The midsurface of the [(c) and (d)] negatively curved surfaces possess a reference metric of constant Gaussian curvature of the opposite sign $K=-0.11 \text{ cm}^{-2}$. Plates (a) and (c) are 0.75 mm thick, whereas plates (b) and (d) are 0.6 mm thick.

bodies that have a rest configuration. The second notable property of our model is that the reference metric is not required to have a vanishing Riemannian curvature tensor, i.e., it may not be immersible in \mathbb{R}^3 (hence, the name “incompatible elasticity theory”). As a result, there exist no rest configurations in which the strains vanish everywhere, hence, the state of frustration. In a second step, a two-dimensional (2D) elasticity theory is derived, using a generalization of the standard Kirchhoff-Love assumptions [13,14]. The end result is an energy functional which depends on surface properties of the midplane of the plate, namely, on the first and second fundamental forms. Like in the classical Föppl–von Kármán theory [15], the energy functional is a sum of a stretching term and a bending term. The bending term is minimized in flat configurations, whereas the stretching term measures, in an L^2 sense, deviations of the 2D surface metric from a prescribed reference metric and vanishes only in surfaces which are isometric immersions of the given 2D metric. The lack of immersibility of the three-dimensional metric manifests in the lack of a planar stretching-free configuration.

At this stage, we have a model, which we believe to be applicable to a large variety of physical and biological systems, whose properties are governed by essentially two-dimensional shaping mechanisms. In [1], a single application was demonstrated for the case of an unconstrained thin plate, whose two-dimensional reference metric is that of a punctured spherical cap. A buckling transition was shown to occur at a critical plate thickness.

In this paper, we study two behaviors exhibited by unconstrained non-Euclidean plates. First, we study the transitions from flat to buckled equilibrium states as the plate thickness crosses a critical value—the buckling threshold. We derive an explicit expression for the critical thickness in terms of the stress field in the planar configuration. An immediate implication is that the plane-stress solution always becomes unstable for sufficiently thin plates provided that it is not trivial, i.e., that the stress is not identically zero. We apply this analysis to three reference geometries of constant Gaussian curvature of different types—an elliptic, a flat, and a hyperbolic metric. We show that the buckling transition may be either continuous (supercritical) or discontinuous (subcritical). In particular, we show that the buckling threshold may deviate significantly from the so-called crossover point, which is based on a balance between the plane-stress energy and the energy that minimizes the Willmore functional. We show an example in which the crossover thickness underestimates the buckling threshold by more than 1 order of magnitude.

Second, we analyze the equilibrium configurations and energies in the limit where the plate thickness tends to zero. We show that if a limit configuration exists, then it is the minimizer of the bending content among all configurations with zero stretching content, i.e., the Willmore energy minimizer among all isometric immersion of the 2D reference metric [16]. For a small but finite thickness, deviations from isometric immersions are more pronounced near the free boundary of the domain, forming a boundary layer, which we obtain in explicit form. In particular, the size of this boundary layer is found to scale with the square root of the plate thickness.

II. THEORY OF NON-EUCLIDEAN PLATES

In this section, we briefly review the modeling of non-Euclidean plates first described in [1]. The starting point is a three-dimensional covariant elasticity theory based on the principles of hyperelasticity [10]: the elastic energy is a volume integral over an energy density, which depends only on (i) the local value of the metric tensor and (ii) local characteristics of the material that are independent of the configuration (the use of the metric tensor, rather than the deformation, as primitive variable was originally proposed by Antman [17] and was recently advocated by Ciarlet and co-workers [11,18–21]).

Let $\Omega \subset \mathbb{R}^3$ be an elastic body endowed with a set of material curvilinear coordinates $\mathbf{x} = (x^1, x^2, x^3) \in \mathcal{D} \subset \mathbb{R}^3$. Let \mathbf{r} denote the mapping from the domain of parametrization \mathcal{D} into $\Omega - \mathbf{r}(\mathbf{x})$ is called the configuration—then the induced Euclidean metric is $g_{ij} = \partial_i \mathbf{r} \cdot \partial_j \mathbf{r}$, where $\partial_i = \partial / \partial x^i$. Our model assumes the existence of a reference metric $\bar{g}_{ij}(\mathbf{x})$, such that the elastic energy density vanishes at a point \mathbf{x} if and only if the actual metric coincides with the reference metric at that point $g_{ij}(\mathbf{x}) = \bar{g}_{ij}(\mathbf{x})$. While the reference metric is required to satisfy the properties of a metric—it is symmetric positive definite—it is not necessarily immersible in \mathbb{R}^3 , hence the name of the theory as “incompatible three-dimensional elasticity.”

The strain tensor is defined as half the deviation of the metric from the reference metric,

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \bar{g}_{ij}).$$

It coincides with the Green-Saint Venant strain tensor in the case where there exists a rest configuration and the curvilinear coordinates form a Cartesian parametrization in the rest configuration, i.e., when $\bar{g}_{ij} = \delta_{ij}$. For small deviations of the metric from the reference metric, the energy functional is truncated at the first nontrivial term, i.e., it is quadratic in the strain tensor, yielding

$$E = \int_{\mathcal{D}} w(g) \sqrt{|\bar{g}|} dx^1 dx^2 dx^3, \quad (1)$$

where

$$w(g) = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl},$$

and

$$A^{ijkl} = \lambda \bar{g}^{ij} \bar{g}^{kl} + \mu (\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk}), \quad (2)$$

where λ, μ are the Lamé coefficients.

Comments.

(1) We adopt the Einstein summation convention whereby repeated indices imply summation.

(2) Latin lowercase characters $i, j, \dots = 1, 2, 3$ are used to denote indices of three-dimensional tensors. We will use below greek characters $\alpha, \beta, \dots = 1, 2$ to denote indices of two-dimensional tensors. For any tensor a_{ij} , $|a|$ denotes its determinant.

(3) The tensor \bar{g}^{ij} is the tensor reciprocal to \bar{g}_{ij} . The raising and lowering of indices is only defined with respect to the reference metric. For example, the tensor g^{ij} is defined as $\bar{g}^{ik}\bar{g}^{jl}g_{kl}$ and not as the reciprocal of g_{ij} , which we denote by $(g^{-1})^{ij}$.

(4) The volume element in Eq. (1) is determined by the reference metric rather than the actual metric. Note that it is not *a priori* clear whether the volume element should be derived from the reference metric or from the actual metric. In any case, the difference between the two choices is of higher order in the strain.

(5) The structure (2) of the elastic tensor is imposed by the assumption of spatial isotropy.

(6) In standard (or “compatible”) nonlinear elasticity, the energy density is sometimes written in terms of the Euclidean distance of the deformation gradient $\nabla \mathbf{r}$ from the group of proper rotations $SO(3)$. In the same spirit, the energy density in an incompatible elasticity theory may be expressed in terms of

$$\text{dist}(\nabla \mathbf{r}, \mathcal{F}_{\bar{g}}),$$

where $\mathcal{F}_{\bar{g}}$ is the set of matrices R such that $R^T R = \bar{g}$.

(7) In summary, given the reference metric \bar{g} , the elastic problem is formulated as follows: find the metric g that minimizes the energy functional (1), subject to the constraint that it is embeddable, in particular, that the corresponding Riemann curvature tensor vanishes.

With a three-dimensional elasticity theory in hand, we focus the attention on platelike structures. We define a plate to be an elastic body for which there exists a parametrization in which the reference metric takes the form

$$\bar{g}_{ij} = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} & 0 \\ \bar{g}_{21} & \bar{g}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

with \bar{g}_{ij} independent of x^3 . The plate is called *even* if $\mathcal{D} = \mathcal{S} \times [-t/2, t/2]$ with $\mathcal{S} \subset \mathbb{R}^2$ and the thickness t is a constant and *thin* if t is much smaller than all other dimensions. We identify the two-dimensional tensor $\bar{g}_{\alpha\beta}$ as the metric tensor of a surface. By assumption, it is constant across the plate thickness. It is easy to see that the three-dimensional reference metric (3) is immersible in \mathbb{R}^3 if and only if the two-dimensional reference metric $\bar{g}_{\alpha\beta}$ has a zero Gaussian curvature.

To derive a reduced two-dimensional energy density in terms of the midsurface configuration, we used an adaptation of the Kirchhoff-Love assumptions [13,14]. We first assume that the stress is parallel to the midsurface and then that $\varepsilon_{i3} = 0$ (the order in which the two assumptions are used is essential). Integrating the energy functional (1) over the thin direction, it takes straightforward manipulations to derive an energy functional which depends on the first and second fundamental forms of the midsurface. Specifically, let $\mathbf{f}(x^1, x^2) = \mathbf{r}(x^1, x^2, 0)$ be the immersion of the midsurface then

$$g_{\alpha\beta} = \partial_{\alpha} \mathbf{f} \cdot \partial_{\beta} \mathbf{f}, \quad h_{\alpha\beta} = \partial_{\beta} \mathbf{f} \cdot \mathcal{N},$$

are the first and second fundamental forms, where \mathcal{N} is the unit vector normal to the midsurface. The energy functional is given by

$$W = tE_S + t^3E_B, \quad (4)$$

where

$$E_S = \int_{\mathcal{S}} w_S dS \quad E_B = \int_{\mathcal{S}} w_B dS$$

are called the stretching and bending contents,

$$w_S = \frac{1}{8} \mathcal{A}^{\alpha\beta\gamma\delta} (g_{\alpha\beta} - \bar{g}_{\alpha\beta})(g_{\gamma\delta} - \bar{g}_{\gamma\delta}),$$

$$w_B = \frac{1}{24} \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta}, \quad (5)$$

are their respective densities, where

$$\mathcal{A}^{\alpha\beta\gamma\delta} = \frac{Y}{2(1+\nu)} \left(\frac{2\nu}{1-\nu} \bar{g}^{\alpha\beta} \bar{g}^{\gamma\delta} + \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta} \bar{g}^{\beta\gamma} \right),$$

and $dS = \sqrt{|\bar{g}|} dx^1 dx^2$ is the infinitesimal surface element. The coefficients Y and ν are Young’s modulus and the Poisson ratio, which can be related to the Lamé coefficients. The elastic energy is positive definite for constant values of $Y > 0$ and $-1 \leq \nu < \frac{1}{2}$.

Comments.

(1) Like in the classical Föppl–von Kármán and Koiter theories, the energy functional is a sum of (i) a stretching energy, which scales with the plate thickness and attains a minimum for an isometric immersion of the midplane surface, and (ii) a bending energy, which scales like the third power of the thickness and attains a minimum for flat configurations. The equilibrium configuration is the minimizer of the sum of both stretching and bending energies.

(2) Rather than working with the energy W given by Eq. (4), we will work with the energy-per-unit thickness,

$$E = \frac{W}{t} = E_S + t^2 E_B. \quad (6)$$

Henceforth, we will refer to E as “the energy.” Thus the stretching energy is t independent whereas the bending energy scales with t^2 . Obviously, both W and E have the same minimizer. In addition, we rescale the units of energy by a factor of $Y/(1+\nu)$, such that the tensor $\mathcal{A}^{\alpha\beta\gamma\delta}$ takes the final form

$$\mathcal{A}^{\alpha\beta\gamma\delta} = \frac{\nu}{1-\nu} \bar{g}^{\alpha\beta} \bar{g}^{\gamma\delta} + \frac{1}{2} (\bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta} \bar{g}^{\beta\gamma}).$$

(3) A different derivation of an elastic functional similar to Eq. (6) may be found in [12]. Equation (6) may be identified as the elastic energy in the Koiter shell model when the “reference bending tensor” $\bar{h}_{\alpha\beta}$ is set to zero [22].

(4) With the above rescaling, the stretching and bending density contents can be written in the more compact form

$$w_S = \frac{\nu}{8(1-\nu)} [\text{tr}(\bar{g}^{-1}g - I)]^2 + \frac{1}{8} \text{tr}[(\bar{g}^{-1}g - I)^2],$$

$$w_B = \frac{\nu}{24(1-\nu)} [\text{tr}(\bar{g}^{-1}h)]^2 + \frac{1}{24} \text{tr}[(\bar{g}^{-1}h)^2].$$

(5) The energy functional is expressed in terms of the first two fundamental forms $g_{\alpha\beta}$ and $h_{\alpha\beta}$ of the midsurface. The two forms are not independent. They must satisfy the three Gauss-Mainardi-Codazzi compatibility conditions,

$$\partial_\epsilon h_{\alpha\beta} - \partial_\beta h_{\alpha\epsilon} = \Gamma_{\alpha\epsilon}^\gamma h_{\gamma\beta} - \Gamma_{\alpha\beta}^\gamma h_{\gamma\epsilon},$$

$$h_{\alpha\epsilon} h_{\beta\eta} - h_{\alpha\beta} h_{\epsilon\eta} = g_{\delta\eta} (\partial_\beta \Gamma_{\alpha\epsilon}^\delta - \partial_\epsilon \Gamma_{\alpha\beta}^\delta + \Gamma_{\alpha\epsilon}^\gamma \Gamma_{\gamma\beta}^\delta - \Gamma_{\alpha\beta}^\gamma \Gamma_{\gamma\epsilon}^\delta),$$

where the Christoffel symbols are given by

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} (g^{-1})^{\gamma\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}).$$

(6) The two-dimensional stress and moment tensors are defined as

$$s^{\alpha\beta} = \mathcal{A}^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}, \quad m^{\alpha\beta} = \frac{t^2}{12} \mathcal{A}^{\alpha\beta\gamma\delta} h_{\gamma\delta},$$

so that

$$w_S + t^2 w_B = \frac{1}{2} s^{\alpha\beta} \varepsilon_{\alpha\beta} + \frac{1}{2} m^{\alpha\beta} h_{\alpha\beta}.$$

(7) A surface $f(x^1, x^2)$ will be called an isometric immersion if the two-dimensional metric $g_{\alpha\beta}$ coincides with the two-dimensional reference metric $\bar{g}_{\alpha\beta}$, i.e., if the stretching energy is zero. In the case of an isometric immersion, the bending content density w_B can be identified with the density of the Willmore functional,

$$w_W = \frac{1}{24} \left(\frac{4H^2}{1-\nu} - 2K \right),$$

where H and K are the mean and Gaussian curvatures of the surface [16]. Note that since K is an isometric invariant, its value is prescribed by the reference metric.

(8) In summary, the two-dimensional elastic problem is defined as follows. Given the two-dimensional reference metric $\bar{g}_{\alpha\beta}$, find a symmetric positive definite tensor field $g_{\alpha\beta}$, and a symmetric tensor field $h_{\alpha\beta}$ that together minimize the energy functional (6) subject to the constraint that the Gauss-Mainardi-Codazzi equations are satisfied.

III. INFINITELY THIN PLATE LIMIT

In many applications, the elastic body is thin to an extent that the equilibrium configuration (of its midsurface) remains practically unchanged upon further thinning. In other words, we identify an asymptotic regime, which we may call the infinitely thin plate limit, which in our model corresponds to the limit $t \rightarrow 0$. It is important to stress that there are also opposite cases, where the thinner the body is, the more convoluted the equilibrium configuration is, with no evidence

that a $t \rightarrow 0$ limit exists (e.g., in [23], a torn plastic sheet exhibits a self-similar shape, whose cut-off scale is comparable to the thickness of the sheet).

Under the assumption that \bar{g} admits an isometric embedding of finite bending content and that a $t \rightarrow 0$ (weak) limit configuration exists, we show in Appendix A that the limit is a minimizer of the Willmore functional among all isometric embeddings.

The first assumption that the bending content is finite is nontrivial. If it does not hold then a limit configuration may not exist. We expect, however, the second assumption, regarding the existence of a (weak) limit, to become eventually superfluous; yet further analysis is required before this assumption can be relaxed.

IV. BUCKLING TRANSITION

A. Plane-stress solution

A configuration is a flat surface if $h_{\alpha\beta} = 0$. The configuration $f(x^1, x^2)$ that minimizes the elastic energy under the constraint that the surface be flat is called the *plane-stress solution*. It is the minimizer of the stretching content, which is given by

$$E = \frac{1}{2} \int_S s^{\alpha\beta} \varepsilon_{\alpha\beta} dS = \frac{1}{8} \int_S \mathcal{A}^{\alpha\beta\gamma\delta} (g_{\alpha\beta} - \bar{g}_{\alpha\beta}) (g_{\gamma\delta} - \bar{g}_{\gamma\delta}) dS,$$

with respect to all flat metrics $g_{\alpha\beta}$. To find the minimizer, we consider an in-plane perturbation of the surface,

$$f \mapsto f + v^\gamma \partial_\gamma f.$$

The reason we perturb the configuration rather than the metric is that the three components of the configuration are independent, whereas the three entries of the metric tensor are constrained by the Gauss-Mainardi-Codazzi relations. The corresponding variation in the metric is

$$\delta g_{\alpha\beta} = \partial_\alpha f \cdot \partial_\beta (v^\gamma \partial_\gamma f) + \partial_\beta f \cdot \partial_\alpha (v^\gamma \partial_\gamma f) + \mathcal{O}(v^2).$$

Using the fact that $\partial_\beta \partial_\alpha f \cdot \partial_\gamma f = \Gamma_{\alpha\beta\gamma}^\eta g_{\eta\gamma}$, the energy variation is

$$\delta E = \int_S s^{\alpha\beta} [g_{\alpha\gamma} (\partial_\beta v^\gamma) + \Gamma_{\beta\gamma}^\eta g_{\eta\alpha} v^\gamma] dS + \mathcal{O}(v^2).$$

Integrating by parts, requiring the first variation to vanish for any perturbation v^γ (and using the identity $\partial_\alpha g_{\beta\gamma} = \Gamma_{\alpha\beta\gamma}^\eta g_{\eta\gamma} + \Gamma_{\alpha\gamma}^\eta g_{\eta\beta}$), the Euler-Lagrange equations are

$$\frac{1}{\sqrt{|\bar{g}|}} \partial_\beta (\sqrt{|\bar{g}|} s^{\eta\beta}) + \Gamma_{\alpha\beta}^\eta s^{\alpha\beta} = 0, \quad (7)$$

with boundary conditions $s^{\alpha\beta} n_\beta = 0$, where n_β is the outward unit vector tangent to the plate and normal to its boundary. We refer to Eq. (7) as the plane-stress *membrane equations*. Note that the plane-stress equations do not depend on the plate thickness t , which only comes into play when there is a competition between stretching and bending energies.

Comment. Equation (7) is the Euler-Lagrange equation associated with the energy functional (6) when $t=0$. It is expected to hold in the limit $t \rightarrow 0$ even for nonflat configura-

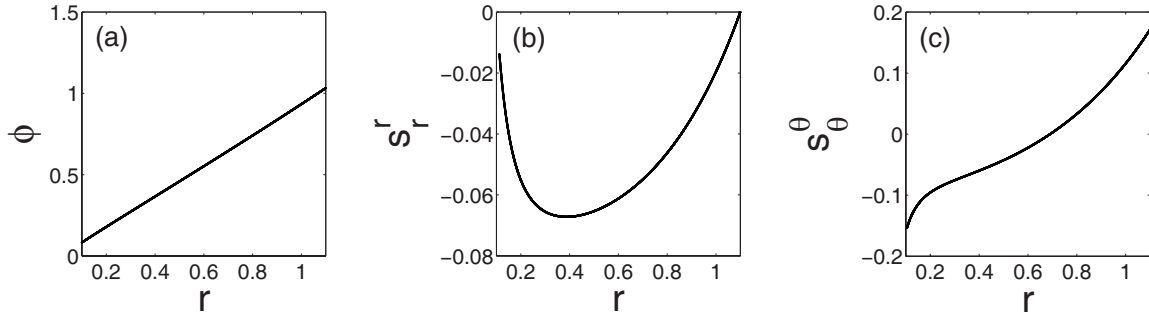


FIG. 2. (a) Plane-stress solution $\phi(r)$ for the elliptic metric (11) with Gaussian curvature $K=1$. [(b) and (c)] The corresponding principal stresses $s_r^r(r)$ and $s_\theta^\theta(r)$.

rations. In general, Eq. (7) constitutes two equations for three unknown functions (the three components of the metric tensor $g_{\alpha\beta}$), i.e., the system is under determined. In the present case, the third equation which removes this under determination is that the curvatures be identically zero.

Examples. In the following examples, we consider for simplicity the case of a vanishing Poisson ratio ($\nu=0$), hence $A^{\alpha\beta\gamma\delta} = \frac{1}{2}(\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta}\bar{g}^{\beta\gamma})$. Denote $x^1=r$ and $x^2=\theta$ and consider a reference metric in semigeodesic parametrization of the form,

$$\bar{g}_{\alpha\beta}(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & \Phi^2(r) \end{pmatrix},$$

with $\Phi(r)$ yet to be specified. The domain of parametrization is

$$(r, \theta) \in [r_{\min}, r_{\max}] \times [0, 2\pi),$$

with periodicity in the θ axis, so that the topology of the body is that of a punctured disk.

The equilibrium configuration is expected to preserve the axis symmetry of the intrinsic geometry; hence we seek plane-stress solutions of the form

$$f(r, \theta) = [\phi(r)\cos\theta, \phi(r)\sin\theta, 0].$$

Elementary calculations show that the resulting two-dimensional metric is

$$g_{\alpha\beta} = \begin{pmatrix} (\phi')^2 & 0 \\ 0 & \phi^2 \end{pmatrix},$$

where $\phi' = d\phi/dr$, from which we derive the two-dimensional stress tensor,

$$s^{\alpha\beta} = \frac{1}{2}\bar{g}^{\alpha\gamma}(g_{\gamma\delta} - \bar{g}_{\gamma\delta})\bar{g}^{\delta\beta} = \frac{1}{2} \begin{pmatrix} (\phi')^2 - 1 & 0 \\ 0 & (\phi^2/\Phi^2 - 1)/\Phi^2 \end{pmatrix}. \quad (8)$$

Finally, the Christoffel symbols are given by

$$\Gamma_{\alpha\beta}^r = \begin{pmatrix} \phi''/\phi' & 0 \\ 0 & -\phi/\phi' \end{pmatrix}, \quad \Gamma_{\alpha\beta}^\theta = \begin{pmatrix} 0 & \phi'/\phi \\ \phi'/\phi & 0 \end{pmatrix}, \quad (9)$$

hence the resulting plane-stress equation is

$$\frac{d}{dr}\{\Phi\phi'[(\phi')^2 - 1]\} = \frac{\phi}{\Phi} \left(\frac{\phi^2}{\Phi^2} - 1 \right), \quad (10)$$

with boundary conditions $\phi'(r_{\min}) = \phi'(r_{\max}) = 1$.

We solve the plane-stress (10) for three families of metrics:

(1) A family of elliptic metrics,

$$\Phi(r) = \frac{1}{\sqrt{K}} \sin\sqrt{Kr}, \quad (11)$$

where $K > 0$ is the constant Gaussian curvature of the reference metric. Although such a metric is consistent with an infinite set of immersions, the immersion that minimizes the Willmore functional is easily identified—it is a (punctured) spherical cap.

(2) A family of conical flat metrics,

$$\Phi(r) = \alpha r, \quad (12)$$

with $\alpha < 1$. Here, the isometric immersion that minimizes the Willmore functional has the form of a truncated cone (a circular frustum). Note that although the reference metric is flat, all isometric immersions have nonzero bending energy due to the topological constraint (periodicity in the θ axis).

(3) A family of hyperbolic metrics,

$$\Phi(r) = \frac{1}{\sqrt{-K}} \sinh\sqrt{-Kr}, \quad (13)$$

where $K < 0$ is the constant Gaussian curvature. Unlike the two former cases, the minimizer of the Willmore functional among all isometric embeddings is not known explicitly; yet, it is known that isometric embeddings with finite bending content do exist [24].

The plane-stress solutions are shown in Figs. 2–4 for the domain $0.1 \leq r \leq 1.1$. The solutions were obtained by a simple shooting procedure, with a fourth-order adaptive ordinary differential equation (ODE) solver. For each metric, we plot the solution $\phi(r)$ along with the spatial profile of the stress components $s_r^r(r)$ and $s_\theta^\theta(r)$ given by Eq. (8) up to the lowering of one index (the reason for displaying stress components with mixed upper and lower indices is that only then all the components have the same units, hence can be compared). In the three cases, the solution $\phi(r)$ is close to linear. Significant differences are, however, observed in the stress components.

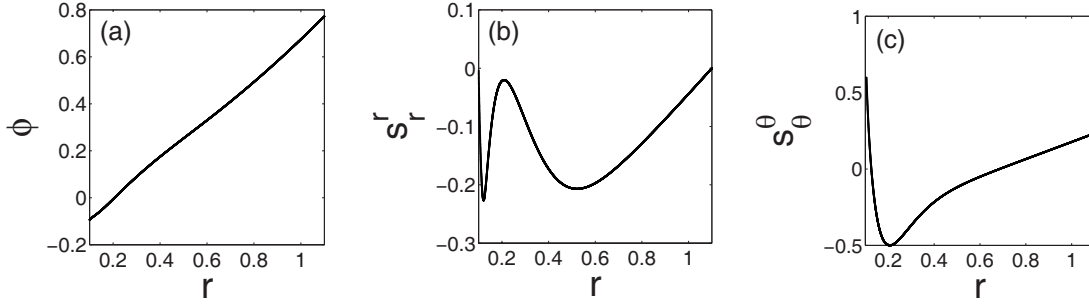


FIG. 3. (a) Plane-stress solution $\phi(r)$ for the flat metric (12) with $\alpha=0.58$. [(b) and (c)] The corresponding principal stresses $s'_r(r)$ and $s''_\theta(r)$.

For the elliptic metric (Fig. 2), the body is in a state of compression along the r direction ($s'_r < 0$), whereas it is compressed in the θ direction near the inner radius and stretched along the θ direction near the outer radius. For the flat metric (Fig. 3), the situation is similar with compression everywhere along the radial direction, while the angular stress switches from extension in the vicinity of the inner boundary to compression at larger radii and again extension in the vicinity of the outer boundary. Finally, for the hyperbolic metric (Fig. 4), the radial stress is everywhere positive (i.e., in a state of extension), whereas the angular stress is in a state of extension very close to the inner radius and in a state of compression at large enough distances from the center. In Fig. 5, we show two toy models generating hyperbolic and elliptic geometries, which elucidate the behavior of the azimuthal (hoop) stress.

B. Stability analysis and buckling threshold

Let $f(x^1, x^2)$ be the plane-stress configuration. Any small enough perturbation can be decomposed into a sum of in-plane and out-of-plane displacements,

$$\delta f = v^\gamma \partial_\gamma f + w \mathcal{N},$$

where \mathcal{N} is the unit vector normal to the surface. Given such a perturbation, we calculate in Appendix B the variation in the elastic energy,

$$\delta E = \int_S (\delta w_S + t^2 \delta w_B) dS,$$

where the variation in stretching content density δw_S is given by Eq. (B6) and the variation in bending content density δw_B is given by Eq. (B16); for flat surfaces, these expressions simplify considerably as $h_{\alpha\beta} = 0$. Note that the plane-stress solution enters in the energy variation both through the stress $s^{\alpha\beta}$ and through the metric parameters $g_{\alpha\beta}$ and $\Gamma_{\alpha\beta}^\gamma$.

The plane-stress solution is locally stable if the energy variation is positive for every choice of sufficiently small (nontrivial) perturbation. As is well-known, local stability can be determined by considering only the leading-order terms (in powers of v, w) in the energy variation. The defining property of the plane-stress solution is that the terms that are linear in the in-plane perturbation v^γ [the integral of $\delta w_S^{(1,0)}$ in Eq. (B6)] vanish for every choice of v^γ . Thus, to the leading order, the energy variation decomposes into a sum of terms that are quadratic in v and terms that are quadratic in w ,

$$\delta E = \delta E_S^{(2,0)}(v) + [\delta E_S^{(0,2)}(w) + t^2 \delta E_B^{(0,2)}(w)] + \mathcal{O}(v^3, v^2 w, v w^2, w^3),$$

where

$$\delta E_S^{(2,0)}(v) = \frac{1}{2} \int_S \{ s^{\alpha\beta} g_{\gamma\epsilon} (\nabla_\alpha v^\gamma) (\nabla_\beta v^\epsilon) + \mathcal{A}^{\alpha\beta\gamma\delta} g_{\beta\eta} g_{\delta\epsilon} (\nabla_\alpha v^\eta) (\nabla_\gamma v^\epsilon) \} dS,$$

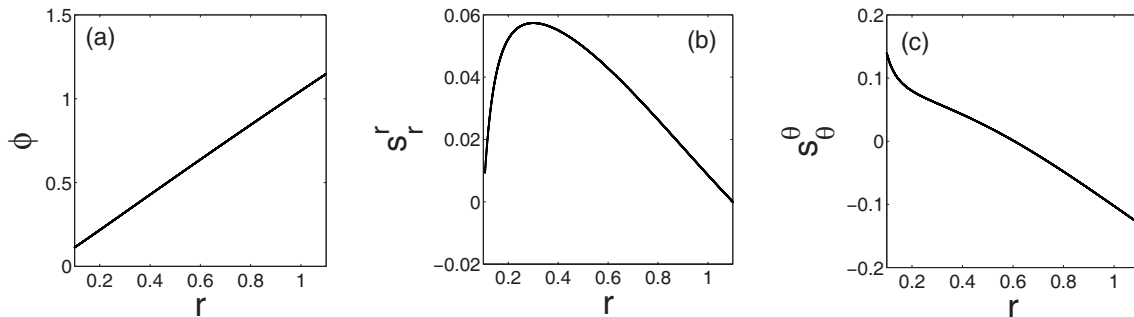


FIG. 4. (a) Plane-stress solution $\phi(r)$ for the hyperbolic metric (13) with Gaussian curvature $K=-1$. [(b) and (c)] The corresponding principal stresses $s'_r(r)$ and $s''_\theta(r)$.

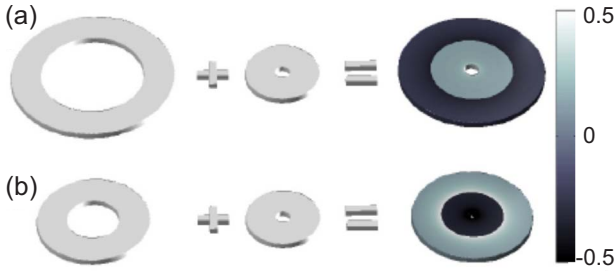


FIG. 5. (Color online) Cartoons of (a) hyperbolic and (b) elliptic plates. In both cases, two punctured disks are “glued” one inside the other and forced to remain planar. In the hyperbolic case, the inner perimeter of the outer disk is too long compared with the outer perimeter of the inner disk. In such a case, the inner disk is stretched azimuthally, while the outer disk is compressed azimuthally. In the elliptic case, the inner perimeter of the outer disk is too short; hence the inner disk is compressed azimuthally, while the outer disk is stretched azimuthally. The color bar on the right represents the azimuthal strain at equilibrium (computed numerically). A similar cartoon was first presented in [25].

$$\delta E_S^{(0,2)}(w) = \int_S \frac{1}{2} s^{\alpha\beta} (\nabla_\alpha w) (\nabla_\beta w) dS,$$

$$\delta E_B^{(0,2)}(w) = \int_S \frac{1}{24} \mathcal{A}^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha w) (\nabla_\delta \nabla_\gamma w) dS \quad (14)$$

[the subscripts (i, j) refer to the power of v and w].

Since, to the leading order, the energy variation is decomposed into a sum of a v -dependent term and a w -dependent term, the minimization can be performed on each component separately. By assumption, the plane-stress solution is the energy minimizer with respect to in-plane perturbations; thus, minimum energy variation is obtained for $v^\gamma = 0$.

It remains to consider the energy variation due to out-of-plane perturbations. The bending term $\delta E_B^{(0,2)}(w)$ is always positive due to the positive definiteness of the tensor $\mathcal{A}^{\alpha\beta\gamma\delta}$ [26]. Whether the stretching term $\delta E_S^{(0,2)}(w)$ is sign definite depends on the plane-stress solution. In fact, if the stress tensor is not *everywhere* positive definite then there exists a perturbation w for which $\delta E_S^{(0,2)}(w)$ is negative, and by taking the plate thickness t sufficiently small, the total-energy variation can be made negative. We have thus recovered the following general result.

Given a reference metric, the plane-stress solution is linearly stable against buckling, independently of the plate thickness, only if the stress is everywhere positive definite. In other words, an infinitely thin plate cannot sustain compression without buckling.

We will now show that the existence of a buckling threshold is always guaranteed, unless the plane-stress solution is trivial, i.e., $s^{\alpha\beta} = 0$ (which in turn occurs only if the reference metric is flat). We start by noting that the plane-stress (7) can be rewritten as

$$\nabla_\beta \left(\frac{\sqrt{|g|}}{\sqrt{|g|}} s^{\alpha\beta} \right) = 0,$$

where ∇_β is the covariant derivative defined in Appendix B. Let now $\chi(x^1, x^2)$ be a scalar field satisfying $\nabla_\beta \nabla_\alpha \chi = 0$ and consider the integral

$$I = \int_S s^{\alpha\beta} (\nabla_\alpha \chi) (\nabla_\beta \chi) dS.$$

The surface element dS is defined in terms of the reference metric. Writing $dS = (\sqrt{|g|}/\sqrt{|g|}) \sqrt{|g|} dx^1 dx^2$, we may now integrate by parts (the covariant derivative satisfies the usual rules of integration by parts provided that the surface element is consistent with the Christoffel symbols), using the boundary conditions $s^{\alpha\beta} n_\beta = 0$,

$$I = \int_S \nabla_\beta \left[\frac{\sqrt{|g|}}{\sqrt{|g|}} s^{\alpha\beta} (\nabla_\alpha \chi) \right] \chi \sqrt{|g|} dx^1 dx^2.$$

Since the covariant derivative satisfies the Leibniz rule for the derivative of products, it follows from the plane-stress equations and the definition of χ that $I = 0$.

Thus, if there exists a scalar function χ that has a nonzero (covariant) gradient and satisfies $\nabla_\alpha \nabla_\beta \chi = 0$ then the fact that $I = 0$ implies that $s^{\alpha\beta}$ is not everywhere positive definite. A simple way to show that such a function does exist is to endow the planar equilibrium state with a Cartesian set of coordinates. Then the covariant derivative reduces into a simple partial derivative, and the function, say, $\chi(x^1, x^2) = x^1$ has the desired property.

We may summarize as follows. A sufficiently thin unconstrained non-Euclidean plate will always buckle unless the plane-stress solution is trivial, i.e., $s^{\alpha\beta} = 0$.

Equation (14) provides a characterization of the critical thickness $t = t_b$ at which buckling first occurs. At criticality $t = t_b$, there exists a nontrivial (i.e., nonuniform) perturbation which to the leading order is marginally unstable, i.e.,

$$\inf_{w \neq \text{const}} \int_S \left\{ \frac{1}{2} s^{\alpha\beta} (\nabla_\alpha w) (\nabla_\beta w) + \frac{t_b^2}{24} \mathcal{A}^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha w) (\nabla_\delta \nabla_\gamma w) \right\} dS = 0,$$

which implies that

$$t_b^2 = \sup_{w \neq \text{const}} \frac{-12 \int_S s^{\alpha\beta} (\nabla_\alpha w) (\nabla_\beta w) dS}{\int_S \mathcal{A}^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha w) (\nabla_\delta \nabla_\gamma w) dS}. \quad (15)$$

By the above analysis, this supremum is guaranteed to be non-negative and zero if and only if $s^{\alpha\beta} = 0$.

Comments.

(1) Equation (15) provides a mean for generating lower bounds for the buckling threshold by choosing appropriate trial functions w .

(2) The energy variation (14) is a quadratic functional of w of the form,

$$\delta E_S^{(0,2)}(w) + t^2 \delta E_B^{(0,2)}(w) = (w, \mathcal{H}w), \quad (16)$$

where (\cdot, \cdot) is the standard inner product on S and \mathcal{H} is a self-adjoint second-order differential operator. Above the buckling threshold, \mathcal{H} is positive definite. The buckling threshold t_b corresponds to the largest t for which \mathcal{H} has a zero eigenvalue. From a numerical point of view, the latter characterization is the easier way for computing the buckling threshold.

Examples. We turn back to the punctured disks considered in the previous subsection. Note that in all three cases, there exist negative stress components; hence, a buckling transition is guaranteed to occur at some finite thickness.

We denote by $\phi(r)$ the solution to the plane-stress (10). For an out-of-plane perturbation $w(r, \theta)$, substituting Eq. (9), we get

$$\begin{aligned} (\nabla_\beta \nabla_\alpha w) &= (\partial_\alpha \partial_\beta w) - \Gamma_{\alpha\beta}^\eta (\partial_\eta w) \\ &= \begin{pmatrix} \phi'(w'/\phi)' & \phi(\dot{w}/\phi)' \\ \phi(\dot{w}/\phi)' & \ddot{w} + \phi(w'/\phi)' \end{pmatrix}, \end{aligned}$$

where we denote by primes derivatives with respect to r and by dots derivatives with respect to θ . Thus,

$$\delta E_S^{(0,2)}(w) = \frac{1}{2} \int_0^{2\pi} \int_{r_{\min}}^{r_{\max}} [s^{rr}(w')^2 + s^{\theta\theta}\dot{w}^2] \Phi dr d\theta,$$

$$\begin{aligned} \delta E_B^{(0,2)}(w) &= \frac{1}{24} \int_0^{2\pi} \int_{r_{\min}}^{r_{\max}} \left\{ (\phi')^2 \left[\left(\frac{w'}{\phi'} \right) \right]^2 + 2 \frac{\phi^2}{\Phi^2} \left[\left(\frac{\dot{w}}{\phi} \right) \right]^2 \right. \\ &\quad \left. + \frac{\phi^2}{\Phi^4} \left(\frac{\dot{w}}{\phi} + \frac{w'}{\phi'} \right)^2 \right\} \Phi dr d\theta, \end{aligned} \quad (17)$$

with s^{rr} and $s^{\theta\theta}$ given by Eq. (8). Due to the periodicity in θ , it is natural to expand the perturbation in Fourier series,

$$w(r, \theta) = a_0(r) + \sqrt{2} \sum_{n=1}^{\infty} a_n(r) \cos n\theta + \sqrt{2} \sum_{n=1}^{\infty} b_n(r) \sin n\theta.$$

Because Eq. (17) is quadratic in w , both terms reduce into a sum over Fourier components,

$$\delta E_S^{(0,2)}(w) = \sum_{n=0}^{\infty} [\delta E_S^n(a_n) + \delta E_S^n(b_n)],$$

$$\delta E_B^{(0,2)}(w) = \sum_{n=0}^{\infty} [\delta E_B^n(a_n) + \delta E_B^n(b_n)],$$

where we define for every function $z=z(r)$,

$$\delta E_S^n(z) = \frac{1}{2} \int_{r_{\min}}^{r_{\max}} \{s^{rr}(z')^2 + s^{\theta\theta}(nz)^2\} \Phi dr,$$

$$\begin{aligned} \delta E_B^n(z) &= \frac{1}{24} \int_{r_{\min}}^{r_{\max}} \left\{ (\phi')^2 \left[\left(\frac{z'}{\phi'} \right) \right]^2 + 2 \frac{\phi^2}{\Phi^2} \left[\left(\frac{nz}{\phi} \right) \right]^2 \right. \\ &\quad \left. + \frac{\phi^2}{\Phi^4} \left(\frac{nz}{\phi} + \frac{z'}{\phi'} \right)^2 \right\} \Phi dr. \end{aligned}$$

The buckling threshold (15) is given by

$$t_b^2 = \sup_{\{a_n, b_n\}} \frac{-\sum_{n=0}^{\infty} [\delta E_S^n(a_n) + \delta E_S^n(b_n)]}{\sum_{n=0}^{\infty} [\delta E_B^n(a_n) + \delta E_B^n(b_n)]}.$$

Corollary 1. Let

$$(t_n^*)^2 = \sup_z \frac{-\delta E_S^n(z)}{\delta E_B^n(z)}.$$

Then it is clearly the case that $t_n^* \leq t_b$ for every n . On the other hand, $t_b \leq \max_n t_n^*$, which together implies that

$$t_b^2 = \max_n \sup_z \frac{-\delta E_S^n(z)}{\delta E_B^n(z)}.$$

Thus, unless the buckling transition is degenerate then the marginally stable perturbation at the bifurcation point involves a single Fourier mode.

Corollary 2. A buckling transition occurs if either s^{rr} or $s^{\theta\theta}$ is somewhere negative. Suppose that $s^{rr}(r) > 0$, i.e., the radial stress is everywhere extensional. It follows that $\delta E_S^0(z) \geq 0$ for every z (every axis symmetric perturbation $n=0$ increases the stretching energy). If $s^{\theta\theta}$ is somewhere negative, then there exist nonaxisymmetric perturbations that reduce the elastic energy. That is, the buckling transition breaks the axial symmetry.

Elliptic geometry. Consider first the elliptic geometry (11) for the same parameters as in Fig. 2. The buckling threshold occurs at $t_b=0.367$ and corresponds to an axis symmetric mode ($n=0$). The critical mode is shown in Fig. 6(a). In Table I, we show the buckling threshold t_b versus the Gaussian curvature K . As expected, the buckling threshold is higher the more curved the surface is.

Flat geometry. Consider next the flat geometry (12) for the same parameters as in Fig. 3. The buckling threshold occurs at $t_b=0.387$, also for an axisymmetric mode. The critical mode is shown in Fig. 6(b).

Hyperbolic geometry. Consider finally the hyperbolic geometry (13) for the same parameters as in Fig. 4. Since $s^{rr} > 0$, it follows that the critical mode must break the polar symmetry. Indeed, the least stable mode, which changes stability at $t_b=0.1845$, has harmonic $n=3$. It is depicted in Fig. 6(c). Note how lower is the buckling threshold for the hyperbolic geometry. Finally, we show in Table II, the buckling threshold t_b versus the Gaussian curvature K .

C. Buckling threshold versus crossover point

Equation (15) expresses the buckling threshold t_b as a supremum over trial normal deflections. As such, it provides

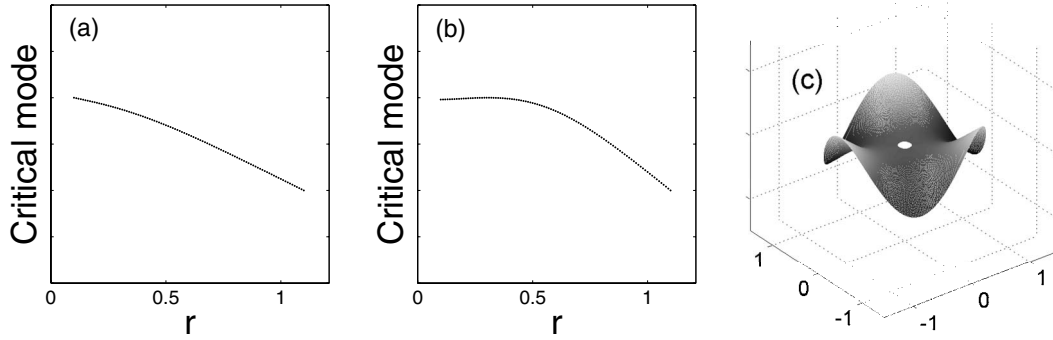


FIG. 6. Critical modes for the (a) elliptic, (b) flat, (c) and hyperbolic geometries. For both elliptic and flat geometries, the critical mode is axisymmetric ($n=0$); hence, we only display a cross section. In the hyperbolic case, the first mode to destabilize is $n=3$.

an easy way to generate lower bounds for the buckling threshold. An approximation often used to estimate the buckling threshold is the so-called crossover point between the lowest-energy isometric immersion and the plane-stress solution. In this section, we show that the crossover point can often yield a significant underestimate to the buckling transition.

The equilibrium configuration is the one that minimizes the energy functional (6). Two upper bounds that correspond to extreme cases are the plane-stress solution, which involves zero bending energy, and the isometric immersion that minimizes the Willmore functional, which involves zero stretching energy. If $g_{\alpha\beta}^{PS}$ denotes the plane-stress metric then

$$E = \frac{1}{8} \int_S \mathcal{A}^{\alpha\beta\gamma\delta} (g_{\alpha\beta}^{PS} - \bar{g}_{\alpha\beta}) (g_{\gamma\delta}^{PS} - \bar{g}_{\delta\gamma}) dS \equiv E_{PS},$$

whereas if $h_{\alpha\beta}^{WF}$ is the second quadratic form that minimizes the Willmore functional (subject to the satisfaction of the Gauss-Mainardi-Codazzi equations with $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$) then the energy reduces into

$$E = \frac{t^2}{24} \int_S \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta}^{WF} h_{\gamma\delta}^{WF} dS \equiv t^2 E_{WF}.$$

Clearly, if the Willmore energy is lower than the plane-stress energy then the plane-stress solution is unstable. This provides a lower bound for the buckling threshold known as the crossover point,

$$t_b \geq \sqrt{\frac{E_{PS}}{E_{WF}}} \equiv t_{co}.$$

Below this thickness, we expect the solution to approach an isometric immersion of minimum energy; thus the energy should be close to E_{WF} . Obviously, in order to evaluate the crossover point one needs to know the minimizer of the Willmore functional, which may be highly nontrivial (it requires,

in particular, the solution of an isometric immersion problem).

Examples. Consider the elliptic and flat geometries (11) and (12), for which the isometric immersion that minimizes the Willmore functional is explicitly known. It is a surface of revolution,

$$f(r, \theta) = [\phi(r)\cos \theta, \phi(r)\sin \theta, \psi(r)]. \quad (18)$$

The corresponding metric is

$$g_{\alpha\beta} = \begin{pmatrix} (\phi')^2 + (\psi')^2 & 0 \\ 0 & \phi^2 \end{pmatrix},$$

hence the isometric immersion satisfies

$$\phi = \Phi, \quad (\phi')^2 + (\psi')^2 = 1.$$

[For the hyperbolic metric (13) $\Phi'(r) > 1$, hence there is no axisymmetric isometric immersion.]

For the elliptic geometry (11) with the same parameters as above, we find

$$E_{PS} = 0.016\ 3, \quad E_{WF} = 0.360\ 9,$$

from which we get $t_{co} = 0.2125$, which is lower than $t_b = 0.367$ by about 40%. In contrast, we obtain for the flat metric (12),

$$E_{PS} = 0.058\ 0, \quad E_{WF} = 75.64,$$

from which we get $t_{co} = 0.027\ 7$, which is lower than $t_b = 0.387$ by more than 1 order of magnitude. This demonstrates that in certain cases, the crossover point may provide a very poor estimate of the buckling threshold. The reason why the discrepancy between t_{co} and t_b may be large is that the buckling transition is a property intrinsic to the plane-stress solution not to isometric immersions.

TABLE I. Buckling threshold t_b versus the Gaussian curvature K for the elliptic geometry (11).

K	0.2	0.4	0.6	0.8	1.0	1.2	1.4
t_b	0.164	0.233	0.285	0.329	0.367	0.401	0.431

TABLE II. Buckling threshold t_b versus the Gaussian curvature K for the hyperbolic geometry (13). In all cases, the critical mode has harmonic $n=3$.

$-K$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
t_b	0.0768	0.110	0.135	0.157	0.175	0.192	0.208	0.222	0.236	0.248

D. Bifurcation analysis

In this section, we analyze the nature of the buckling transition. At $t=t_b$, the plane-stress solution is marginally stable. In particular, there exists a nontrivial perturbation that does not change the elastic energy up to terms that are quadratic in v, w . Specifically,

$$\delta E_S^{(2,0)}(v) = 0 \quad \text{if and only if } v = 0,$$

and there exists a $\tilde{w} \neq 0$ such that

$$\delta E_S^{(0,2)}(\tilde{w}) + t_b^2 \delta E_B^{(0,2)}(\tilde{w})$$

changes sign at $t=t_b$. In fact, as $\delta E_B^{(0,2)}$ can be identified as an inner product [cf. Eq. (16)], it follows that for every \hat{w} ,

$$(\hat{w}, \mathcal{H}\tilde{w}) = 0. \tag{19}$$

Since \tilde{w} is determined up to both additive and multiplicative constants, we will define \tilde{w} to have zero mean and be normalized $\|\tilde{w}\|_2 = 1$, where $\|\cdot\|_2$ is the L^2 norm.

For plate thickness below t_b , the flat configuration is linearly unstable. (Note, however, that the plane-stress solution is a critical point of the energy functional for all values of t ; it only ceases to be a local minimum at t_b). The loss of stability of the flat solution is due to a bifurcation. A branch of stationary solutions with nonzero bending content merges with the plane-stress solution at $t=t_b$. The bifurcation is called supercritical (forward) if the branch of buckled solutions exists for $t \leq t_b$, in which case, as predicted by the bifurcation theory [27], the buckled solutions near t_b are linearly stable. For a supercritical bifurcation, the transition from the plane-stress solution to the buckled solution, as t decreases below t_b , is continuous. The bifurcation is called subcritical (backward) if the branch of buckled solutions exists for $t \geq t_b$. In this case, the buckled solutions near t_b are unstable (this branch of solutions becomes stable after it turns back). A transition to linearly stable solutions occurs discontinuously at $t=t_b$. In particular, discontinuous bifurcations exhibit hysteresis.

To analyze the bifurcation, we need to study the behavior of the energy functional in the vicinity of the bifurcation threshold. Since the plane-stress solution is marginally stable at t_b , terms that contain higher order in v, w must be taken into account.

Let $g_{\alpha\beta}$ and $s_{\alpha\beta}$ be the plane-stress metric and stress and set $t^2 = t_b^2 - \epsilon$ with $\epsilon > 0$ a small parameter. That is, we consider plate thicknesses just below the buckling threshold. By the above discussion, the bifurcation is supercritical if for small ϵ the energy functional has a local minimum for a nontrivial perturbation whose magnitude vanishes as $\epsilon \downarrow 0$. If the bifurcation is subcritical then the stable solution for $\epsilon > 0$ does not converge to the plane-stress solution as $\epsilon \downarrow 0$. Our working hypothesis is that the bifurcation is supercriti-

cal. The analysis will prove us wrong if this is not the case.

Set once again $\delta f = v^\gamma \partial_\gamma f + w \mathcal{N}$. Substituting the variations (B6) and (B16) in stretching and bending content densities, the variation in total energy takes the form

$$\begin{aligned} \delta E = & \delta E_S^{(2,0)}(v) + \delta E_S^{(0,2)}(w) + \delta E_S^{(1,2)}(v, w) + \delta E_S^{(0,4)}(w) \\ & + (t_b^2 - \epsilon) \{ \delta E_B^{(0,2)}(w) + \delta E_B^{(1,2)}(v, w) + \delta E_B^{(0,4)}(w) \} \\ & + \mathcal{O}(v^3, v^2 w, v w^3, w^5), \end{aligned}$$

where $\delta E_S^{(2,0)}(v)$, $\delta E_S^{(0,2)}(w)$, and $\delta E_B^{(0,2)}(w)$ are given by Eq. (14) and

$$\delta E_S^{(1,2)}(v, w) = \frac{1}{2} \int_S A^{\alpha\beta\gamma\delta} g_{\beta\gamma} (\nabla_\alpha v^\eta) (\nabla_\gamma w) (\nabla_\delta w) dS,$$

$$\delta E_S^{(0,4)}(w) = \frac{1}{8} \int_S A^{\alpha\beta\gamma\delta} (\nabla_\alpha w) (\nabla_\beta w) (\nabla_\gamma w) (\nabla_\delta w) dS,$$

$$\delta E_B^{(1,2)}(v, w) = - \frac{1}{12} \int_S A^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha w) (\nabla_\delta \nabla_\gamma v^\eta) (\nabla_\eta w) dS,$$

$$\begin{aligned} \delta E_B^{(0,4)}(w) = & - \frac{1}{24} \int_S A^{\alpha\beta\gamma\delta} (g^{-1})^{\eta\epsilon} (\nabla_\eta w) (\nabla_\epsilon w) (\nabla_\beta \nabla_\alpha w) \\ & \times (\nabla_\delta \nabla_\gamma w) dS. \end{aligned}$$

We are seeking the perturbation that minimizes the energy variation for small $\epsilon > 0$. Since we expect, to the leading order, the minimizer to be proportional to \tilde{w} (the least stable out-of-plane mode at t_b) with a prefactor that vanishes as $\epsilon \rightarrow 0$, we expand the minimizer in a power series in ϵ , whose first terms are

$$v^\gamma = c^2 \tilde{v}^\gamma \epsilon^p + \dots,$$

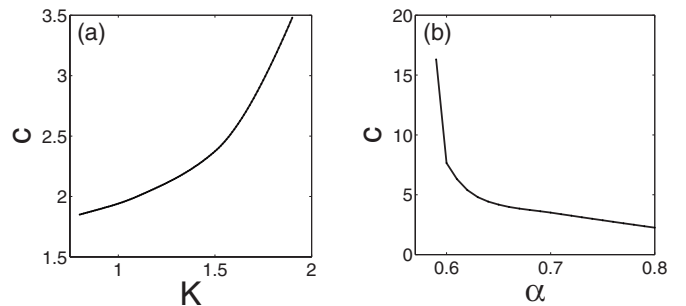


FIG. 7. The coefficient c versus the Gaussian curvature K of the (a) elliptic geometry and the parameter α of the (b) flat geometry.

$$w = c\tilde{w}\epsilon^q + \tilde{\tilde{w}}\epsilon^r + \dots, \quad (20)$$

where the exponents p, q, r and the constant c are yet to be determined. Substituting this expansion into the energy variation, we get

$$\begin{aligned} \delta E = & -c^2[\delta E_B^{(0,2)}(\tilde{w})]\epsilon^{2q+1} + c^4[\delta E_S^{(2,0)}(\tilde{v})]\epsilon^{2p} \\ & + c^4[\delta E_S^{(1,2)}(\tilde{v}, \tilde{w}) + t_b^2 \delta E_B^{(1,2)}(\tilde{v}, \tilde{w})]\epsilon^{p+2q} \\ & + c^4[\delta E_S^{(0,4)}(\tilde{w}) + t_b^2 \delta E_B^{(0,4)}(\tilde{w})]\epsilon^{4q} \\ & + \mathcal{O}(\epsilon^{2r}, \epsilon^{3p}, \epsilon^{5q}, \epsilon^{2p+q}, \epsilon^{p+3q}, \epsilon^{2p+1}, \epsilon^{4q+1}, \epsilon^{p+2q+1}). \end{aligned}$$

The first term on the right-hand side is the quadratic out-of-plane term, which is, as expected, negative for $\epsilon > 0$. For a supercritical bifurcation, it is balanced by the quartic term,

$$c^2 = \frac{\frac{1}{2} \delta E_B^{(0,2)}(\tilde{w})}{\delta E_S^{(2,0)}(\tilde{v}) + [\delta E_S^{(1,2)}(\tilde{v}, \tilde{w}) + t_b^2 \delta E_B^{(1,2)}(\tilde{v}, \tilde{w})] + [\delta E_S^{(0,4)}(\tilde{w}) + t_b^2 \delta E_B^{(0,4)}(\tilde{w})]}.$$

Recall that ϵc is, to the leading order in ϵ , the L^2 norm of the out-of-plane deflection w . If the denominator is negative then the bifurcation is subcritical and the branch of stable buckled solutions cannot be found by a local analysis about the plane-stress solution.

We calculated c for both the elliptic and flat geometries; recall that in both cases, the critical mode is axisymmetric $n=0$. For the elliptic geometry, the bifurcation was found to be supercritical for the whole possible range of curvatures K (for large enough K , the surface is no longer an embedding, as the sphere closes upon itself). For the flat geometry, a transition from supercritical to subcritical bifurcations was found. The bifurcation is supercritical for α in the range $0.58 < \alpha < 1$ and subcritical for $\alpha < 0.58$.

In Fig. 7, we show the value of c versus the Gaussian curvature K of the elliptic geometry (left) and the parameter α of the flat geometry (right). Note that at the transition point from supercritical to subcritical bifurcation $\alpha \approx 0.585$, the coefficient c diverges.

V. BOUNDARY LAYERS IN VERY THIN PLATES

It was shown in Sec. III that provided that a limit equilibrium configuration as $t \rightarrow 0$ exists, it is given by the isometric immersion that minimizes the Willmore functional. How a sequence of equilibrium configurations approaches the Willmore isometry is nontrivial. The convergence is in the Sobolev space $W^{2,2}$ —the space of surfaces with square integrable second (weak) derivatives [28]. This guarantees (by the Sobolev embedding theorem) uniform convergence in the space of once-differentiable embeddings but not in the space of twice-differentiable embeddings. In other words, second derivatives may not converge uniformly.

from which we infer that $2q+1=4q$, i.e., $q=1/2$. Since the term that is quadratic in v is positive, it follows that $2p \geq 2q+1=2$. We may then set $p=1$, with the possible outcome that we obtain $\tilde{v}=0$ (i.e., that the v terms are subdominant).

We proceed to minimize this expression (with all four terms of the order ϵ^2) with respect to the in-plane perturbation \tilde{v} and the constant c . Note that under the normalization choice in Eq. (20), the minimizing \tilde{v} does not depend on c since the two \tilde{v} -dependent terms are proportional to c^4 . The \tilde{v} -dependent terms consist of a positive definite quadratic term and a linear term, which guarantees the existence of a nontrivial minimizer (and, in particular, confirms that $p=1$). Once \tilde{v} has been determined, a minimizing c exists if and only if the sum of the terms proportional to c^4 are positive. Then,

Almost a hundred years ago (see [29], and references therein), it was observed that thin elastic bodies may exhibit boundary layers, which interpolate between a state of minimum stretching content in the bulk and the zero normal traction and zero bending moment conditions at the boundary. Such boundary layers also occur in non-Euclidean plates and turn out to dominate the deviation from an isometry as $t \rightarrow 0$.

Generally speaking, a large thickness implies a bending energy-dominated configuration (i.e., close to flat), whereas a small thickness implies a stretching energy-dominated configuration (i.e., close to an isometry). Whether a thickness t is to be considered as “large” or “small” is determined by comparison with the shortest length scale of the problem, which may vary with position. For every finite t , there exists a distance from the boundary ℓ , with respect to which t cannot be considered small. As a result, we expect a bending energy-dominated behavior in a strip of thickness ℓ near the boundary.

We start with a scaling argument. Let $h_{\alpha\beta}$ be the second fundamental form of an isometric immersion $f(x^1, x^2)$ that minimizes the Willmore functional (the metric is, of course, equal to the reference metric $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$). From the point of view of the bending energy, it would be favorable to have a flat surface $h_{\alpha\beta}=0$; however, the second fundamental form cannot be modified without a modification of the metric, as the two must satisfy the Gauss-Mainardi-Codazzi equations. In particular, the Gaussian curvature is an isometric invariant. From the analysis in Appendix B, assuming that the perturbations do not involve *small-scale features*, we see that the variation in stretching content is quadratic in the perturbation fields v, w , whereas the variation in bending content is linear in v, w , i.e.,

$$\delta E \sim \mathcal{O}(v^2, w^2) + t^2 \mathcal{O}(v, w).$$

Since equilibrium is obtained by a balance of the negative bending contribution and the positive stretching contribution then $v, w \sim \mathcal{O}(t^2)$ and $\delta E \sim \mathcal{O}(t^4)$.

These are, however, bulk considerations, where the energy balance is considered uniformly over the surface. The question is whether the total elastic energy can be reduced by a larger yet *local* change in the bending content density. This is not possible inside the domain because even a local change in the second fundamental form involves a nonlocal change in the metric; hence the gain in stretching energy exceeds the loss in bending energy. The situation may however be different at the boundary.

In a bending-dominated region, we expect the curvatures to deviate from the curvatures associated with the Willmore isometry by $\mathcal{O}(1)$. As curvatures relate to the metric through two differentiations, such a deviation over a strip of width t^p induces a metric deviation of $\mathcal{O}(t^{2p})$. Thus, the variation in the total energy is of the order

$$\delta E \sim \mathcal{O}(t^{4p}) + t^2 \mathcal{O}(1),$$

which yields $p=1/2$.

To make this into a rigorous argument, suppose that $v = \mathcal{O}(t^q)$ and $w = \mathcal{O}(t^r)$, where $q, r > 0$, both varying over a boundary layer whose width scales like t^p , where $p > 0$. Inside the boundary layer, the variation in stretching energy density (B6) is dominated by terms of the order

$$\delta w_S = \mathcal{O}(t^{2q-2p}, t^{2r}, t^{q+r-p}, t^{q+2r-3p}, t^{3r-2p}, t^{4r-4p}).$$

Note that this contribution is always positive. On the other hand, the variation in bending energy density (B14), which can become negative, is dominated for small t by terms of the order

$$t^2 \delta w_B = \mathcal{O}(t^{q-p+2}, t^{r-2p+2}).$$

The exponents q, r, p are determined such as to maximize the change in the (negative) bending energy, without the (positive) stretching energy becoming dominant. That is, if we define

$$e_S = \min(2q - 2p, 2r, q + r - p, q + 2r - 3p, 3r - 2p, 4r - 4p),$$

$$e_B = \min(q - p + 2, r - 2p + 2),$$

then we need to choose q, r, p such to minimize e_B subject to the constraint that $e_S \geq e_B$. It can be shown that the optimal choice satisfies $r=1$, $p=1/2$, and $q \geq 3/2$. That is, the width of the boundary layer is expected to scale like the square root of the plate thickness. To minimize the gain in stretching content, we expect a balance between the $\nabla_\alpha v^\beta$ and w terms, which yields $q=3/2$.

Assuming these scaling exponents, we study the structure of the boundary layer. We consider, as before, a perturbation, which we decompose as

$$\delta \mathbf{f} = v^\gamma \partial_\gamma \mathbf{f} + w \mathcal{N}.$$

Consider now a local parametrization $\mathbf{f}: [0, \ell^1] \times [0, \ell^2] \rightarrow \mathbb{R}^3$ of the surface, such that the parametric line $x^1=0$ co-

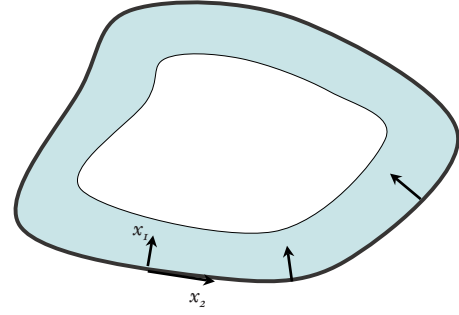


FIG. 8. (Color online) Local parametrization of an annulus bounded by a boundary of the domain.

incides with a boundary of the surface, with the positive x^1 axis inside the sample. Moreover, the parametrization of the unperturbed surface is semigeodesic, i.e., $g_{11}=1$, $g_{12}=g_{21}=0$, and $g_{22}=\phi^2$; such a parametrization is always possible. One may also set $g_{22}=1$ along the boundary (see Fig. 8).

Since we expect a boundary layer of size \sqrt{t} , we stretch the positive x^1 axis accordingly by introducing a rescaled coordinate,

$$X^1 = \frac{x^1}{\sqrt{t}},$$

and rescale the perturbations v^α, w , such that the new variables *and their derivatives* are all on order of 1,

$$V^\gamma(X^1, x^2) = \frac{1}{t^{3/2}} v^\gamma(\sqrt{t} X^1, x^2),$$

$$W(X^1, x^2) = \frac{1}{t} w(\sqrt{t} X^1, x^2). \quad (21)$$

By setting, say, $\ell^1 \sim \mathcal{O}(t^{1/4})$ we have a situation where, as $t \rightarrow 0$, the local coordinates (x^1, x^2) parametrize a shrinking annulus which converges to the boundary; whereas in the rescaled coordinates (X^1, x^2) , the range of X^1 in the positive direction tends to infinity. We are going to show the existence of a perturbation of such structure that reduces the total elastic energy.

We then evaluate the variation in energy content densities inside the boundary layer, i.e., at points $x^1 = \sqrt{t} X^1$, with $X^1 \sim \mathcal{O}(1)$. To the leading order, the unperturbed metric and the Christoffel symbols are given by their values at the boundary and covariant derivatives coincide with partial derivatives. Since $\bar{g}_{\alpha\beta} = g_{\alpha\beta}$, we also have

$$A^{\alpha\beta\gamma\delta} = \frac{\nu}{1-\nu} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{1}{2} (\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) + \mathcal{O}(t^{1/2}). \quad (22)$$

Substituting the rescaled variables (21) into the variations (B6) and (B14) in stretching and bending content densities, we get

$$\begin{aligned} \delta w_S/t^2 &= \frac{1}{2} \mathcal{A}^{1\beta 1\delta} (\partial_1 V_\beta) (\partial_1 V_\delta) + \frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta} W^2 \\ &\quad - \mathcal{A}^{1\beta\gamma\delta} h_{\gamma\delta} (\partial_1 V_\beta) W + \frac{1}{2} \mathcal{A}^{1111} (\partial_1 V^1) (\partial_1 W)^2 \\ &\quad - \frac{1}{2} \mathcal{A}^{\alpha\beta 11} h_{\alpha\beta} (\partial_1 W)^2 W + \frac{1}{8} \mathcal{A}^{1111} (\partial_1 W)^4 + \mathcal{O}(t^{1/2}), \end{aligned}$$

and

$$\delta w_B = \frac{1}{12} \mathcal{A}^{\alpha\beta 11} h_{\alpha\beta} (\partial_1 \partial_1 W) + \frac{1}{24} \mathcal{A}^{1111} (\partial_1 \partial_1 W)^2 + \mathcal{O}(t^{1/2}).$$

Substituting also expression (22) for the elastic tensor, we end up with

$$\begin{aligned} \delta w_S/t^2 &= \frac{1}{4} [(\partial_1 V^2) - 2h_{12}W]^2 + \frac{1}{2} h_{22}^2 W^2 + \frac{\nu}{8(1-\nu)} [2(\partial_1 V^1) \\ &\quad - 2(h_{11} + h_{22})W + (\partial_1 W)^2]^2 + \frac{1}{8} [2(\partial_1 V^1) - 2h_{11}W \\ &\quad + (\partial_1 W)^2]^2 + \mathcal{O}(t^{1/2}), \end{aligned}$$

$$\delta w_B = \frac{1}{24(1-\nu)} [2(h_{11} + \nu h_{22})(\partial_1 \partial_1 W) + (\partial_1 \partial_1 W)^2] + \mathcal{O}(t^{1/2}).$$

The variation in the stretching content density is a sum of squares. The optimal choice of V^2 is the one that makes the first term vanish, namely,

$$(\partial_1 V^2) = 2Wh_{12}.$$

Similarly, the optimal choice for V^1 satisfies

$$2(\partial_1 V^1) + (\partial_1 W)^2 = 2(h_{11} + \nu h_{22})W,$$

so that the variational problem reduces into one for W only,

$$\left. \frac{\delta w_S}{t^2} \right|_{V^1, V^2 \text{ optimal}} = \frac{1}{2} (1 + \nu) h_{22}^2 W^2 + \mathcal{O}(t^{1/2}).$$

Omitting the $\mathcal{O}(t^{1/2})$ terms, the resulting Euler-Lagrange equations are

$$\frac{\delta^4 W}{\partial (X^1)^4} + 12(1 - \nu^2) h_{22}^2 W = 0,$$

with boundary conditions

$$\frac{\delta^2 W}{\partial (X^1)^2} = -(h_{11} + \nu h_{22}), \quad \frac{\delta^3 W}{\partial (X^1)^3} = 0,$$

at $X^1=0$ and $W(\infty, x^2)=0$. The solution is

$$W(X^1, 2) = -\frac{h_{11} + \nu h_{22}}{2\lambda^2} e^{-\lambda X^1} (\cos \lambda X^1 - \sin \lambda X^1), \quad (23)$$

where

$$\lambda = [3(1 - \nu^2) h_{22}^2]^{1/4}.$$

Reverting to the original unscaled units (x^1, x^2), the out-of-plane perturbation $w(x^1, x^2)$ exhibits a boundary layer whose width ℓ_{BL} is

$$\ell_{BL} = [3(1 - \nu^2)]^{-1/4} \sqrt{t/|h_{22}|}.$$

Substituting the asymptotic boundary layer profile (23) in the leading order expressions for $\delta w_S/t^2$ and δw_B , we get

$$\delta w_S/t^2 = \frac{(h_{11} + \nu h_{22})^2}{24(1-\nu)} e^{-2\lambda X^1} (\cos \lambda X^1 - \sin \lambda X^1)^2,$$

$$\begin{aligned} \delta w_B &= \frac{(h_{11} + \nu h_{22})^2}{24(1-\nu)} e^{-2\lambda X^1} (\cos \lambda X^1 + \sin \lambda X^1) (\cos \lambda X^1 \\ &\quad + \sin \lambda X^1 - 2e^{\lambda X^1}). \end{aligned} \quad (24)$$

At the boundary, to the leading order, curvature in the direction normal to the boundary is given by

$$h_{11} + \partial_1 \partial_1 W = -\nu h_{22}.$$

This leads to the vanishing of the bending moment at the boundary, i.e., $m^{11}=0$, which is one of the boundary conditions associated with the energy minimization variational problem [1]. Note that the normal bending moment along the boundary is proportional to $h_{11} + \nu h_{22}$. If $\nu=0$ and the bending minimizing isometry satisfies $h_{11}=0$ then no correction is needed in order to satisfy the boundary conditions, and we expect no boundary layer to develop. This fact is manifested by the vanishing of W in Eq. (23). For $\nu \neq 0$, however, the curvatures normal and tangent to the boundary have opposite signs, which means that the surface is hyperbolic in the vicinity of the boundary. Finally, our analysis breaks down in the event that $h_{22}=0$, in which case a boundary layer of different nature may emerge.

To validate our results, we plot in Fig. 9 the rescaled deviations in stretching content density $\delta w_S/t^2$ and the deviations in bending content density δw_B versus the rescaled coordinate $(r_{\max} - r)/\sqrt{t}$, for the elliptic geometry (11), with the same parameters as in previous sections, namely, $r_{\min}=0.1$, $r_{\max}=1.1$, $K=1$, and $\nu=0$. The results were obtained by minimizing the full energy functional. We display results for $t=0.01, 0.005, 0.002$, and 0.001 . As expected, the four rescaled curves approximately coincide. These numerically computed curves are compared with our asymptotic expressions (24).

VI. DISCUSSION

A theory of non-Euclidean plates, applicable to thin elastic sheets that do not have a stress-free rest configuration, has recently been proposed in [1]. This paper provides a first mathematical analysis of this model. Two different limits of such plates are analyzed: (i) the buckling transition, and (ii) the occurrence of boundary layers in the limit where the plate thickness tends to zero, and the configuration is expected to converge to the isometric immersion that minimizes the Willmore functional.

We proved a general result, whereby any non-Euclidean plate that does not have a flat stress-free configuration (i.e., whose reference metric has nonzero Gaussian curvature) buckles if the plate is sufficiently thin. The transition from flat to buckled equilibria may be either continuous or discon-

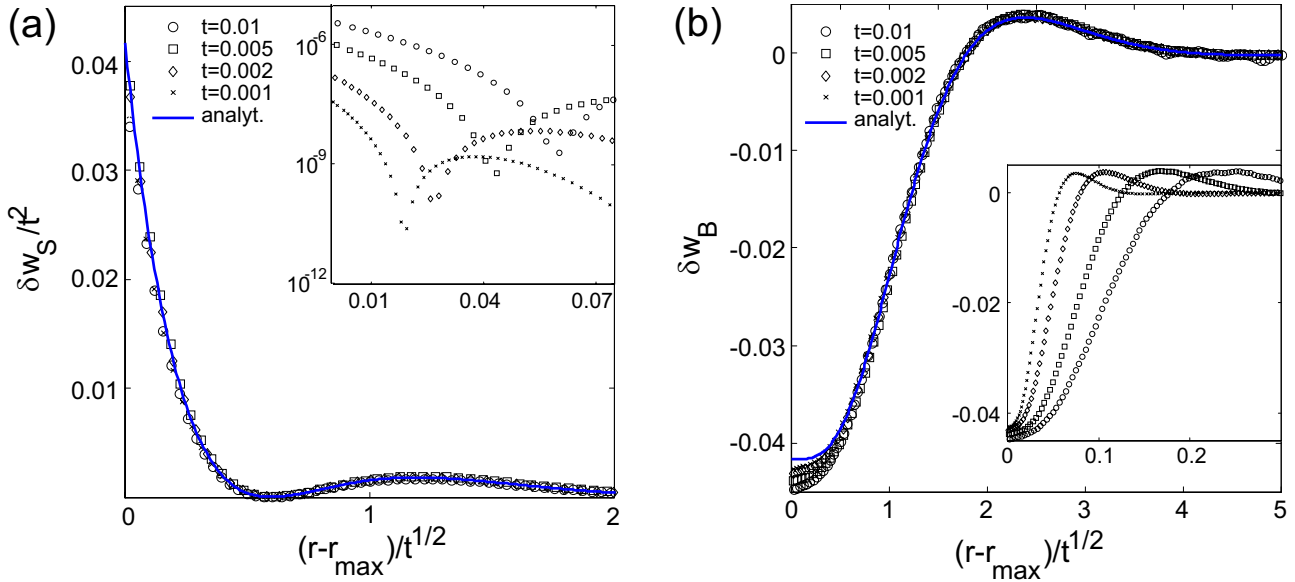


FIG. 9. (Color online) Structure of the boundary layer near the outer boundary $r=1.1$, for the elliptic geometry (11) and the same parameters as in previous sections. (a) The left figure shows the rescaled deviation in stretching content density $\delta w_S/t^2$ versus the rescaled coordinate $(r_{\max}-r)/\sqrt{t}$ calculated for four different thicknesses $h=0.01, 0.005, 0.002$, and 0.001 (symbols). The solid (blue) line is the asymptotic result (24). The inset shows the unscaled stretching content density deviations versus the unscaled coordinate $r_{\max}-r$. Note the logarithmic y scale. (b) The right figure shows the bending content density δw_B versus the rescaled coordinate for the same values of the thickness. The inset shows the unscaled results.

tinuous, depending on the particular system. Instances of both types have been observed.

We showed that in the thin plate regime, the dominant deviation from the Willmore isometry is governed by a bending-dominated boundary layer, whose structure was calculated using a boundary layer analysis. In particular, the width of this boundary layer is determined by both the plate thickness and the tangential curvature at the boundary of the Willmore isometry—it scales like the square root of their product. It would be of interest to observe the occurrence of such boundary layers in experiments, e.g, in the thermoresponsive gels studied in [4], in order to further validate the model as describing thin elastic sheets with no stress-free configuration.

The reported results are of high relevance to the shape formation in growing biological tissues. Spontaneous buckling and wrinkling of non-Euclidean sheets were suggested as a mechanism for shaping leaves that are free of external confinement [2,3,23,30]. These studies were either qualitative or assumed the limit of an infinitely thin sheet. As such, their results are relevant only to selected cases. Recent studies suggest and demonstrate that the mechanical stress field can lead to the differentiation of cells [31–33] and might act as a regulator of tissue growth [34]. These studies emphasize the need to know the mechanical state of a tissue of finite thickness, which undergoes differential growth. Our calculations of buckling threshold, as well as the prebuckling and postbuckling stress distribution within plates, can be integrated into the biological picture as inputs that can affect its future evolution. In particular, the existence of a boundary layer and its scaling predict thickness-dependent localized effects near the boundaries of unconstrained growing tissues.

ACKNOWLEDGMENTS

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APPENDIX A: THE $t \rightarrow 0$ LIMIT

The $t \rightarrow 0$ limit can be approached in two ways. The first possibility is to depart from the three-dimensional model, i.e., the energy functional (1), and analyze the limit of the energy minimizers (or approximate energy minimizers) as $t \rightarrow 0$. To this end, one would hope to be able to use the analytical techniques based on Γ convergence developed in [35–37]. There is, however, an obstacle that prevents the direct application of the abovementioned techniques to the present context. We do not have a reference configuration with respect to which deviations can be measured. Indeed, the analysis in [36] relies heavily on a so-called rigidity property that estimates the distance of the displacement from a rigid rotation in terms of an integral over local distances from rotations.

The second alternative is to depart from the two-dimensional model, i.e., the energy functional (6). For reasons to be clarified below, we work with an energy functional rescaled with the thickness square,

$$F_t = \frac{E}{t^2} = \frac{1}{t^2}E_S + E_B, \quad (\text{A1})$$

where the notation F_t makes the t dependence explicit. Clearly, for every fixed t the functionals E and F_t have the same minimizers. We view F_t as a one-parameter family of functionals defined on the Sobolev space $W^{2,2}(\mathcal{S}; \mathbb{R}^3)$ of surfaces whose second (weak) derivatives are in $L^2(\mathcal{S})$. Since we view every two configurations that differ by a rigid motion as identical, the space of immersions is in fact the quotient space $W^{2,2}(\mathcal{S}; \mathbb{R}^3)$ modulo rigid motions. We denote by $F_t[f]$, $E_S[f]$, and $E_B[f]$ the functionals F_t , E_S , and E_B evaluated at a configuration $\mathbf{f} = \mathbf{f}(x^1, x^2)$. We will also denote by

$$e_t = \inf\{F_t[f]; \mathbf{f} \in W^{2,2}(\mathcal{S}; \mathbb{R}^3)\}$$

the t -dependent greatest lower bounds on the energy.

The two-dimensional elastic problem formulated in the Sec. II assumes the existence of a family of minimizers \mathbf{f}_t^* such that $F_t[\mathbf{f}_t^*] = e_t$. Even if such minimizers do not exist, we can always construct a family of *approximate minimizers* \mathbf{f}_t^* , satisfying,

$$\lim_{t \rightarrow 0} (F_t[\mathbf{f}_t^*] - e_t) = 0.$$

Suppose now that the two-dimensional reference metric $\bar{g}_{\alpha\beta}$ assumes an isometric immersion $\mathbf{f} = \hat{\mathbf{f}}$ with finite bending content. Then for every t ,

$$e_t \leq F_t[\hat{\mathbf{f}}] = E_B[\hat{\mathbf{f}}],$$

i.e., the greatest lower bounds on the energy are uniformly bounded. In particular, it follows that

$$\lim_{t \rightarrow 0} E_S[\mathbf{f}_t^*] = 0,$$

which means that the metrics $g(\mathbf{f}_t^*)$ associated with the family of approximate minimizers converge, in least-square norm, to the reference metric \bar{g} .

The mean-square convergence of the family of metrics, as $t \rightarrow 0$, does not guarantee that the family of (possibly approximately) minimizing configurations has a limit (modulo rigid motions). If, however, such a limit does exist then we show that this limit is an isometric immersion that minimizes the bending content, i.e., the Willmore functional. Specifically, let \bar{g} be a reference metric that assumes a $W^{2,2}$ isometric immersion. Let \mathbf{f}_t^* be a family of approximate minimizers of the functionals F_t . If the family \mathbf{f}_t^* (weakly) converges in $W^{2,2}(\mathcal{S}; \mathbb{R}^3)$, as $t \rightarrow 0$, to a limit \mathbf{f}^* then \mathbf{f}^* is a configuration that minimizes the Willmore functional over all isometric immersions of the reference metric \bar{g} .

To prove this theorem we construct a ‘‘limit functional,’’

$$F_0[\mathbf{f}] = \begin{cases} E_B[\mathbf{f}] & g(\mathbf{f}) = \bar{g} \\ \infty & \text{otherwise,} \end{cases}$$

and show that the functionals $F_t \Gamma$ converge to F_0 , as $t \rightarrow 0$, with respect to the weak $W^{2,2}$ topology. It then follows that every converging sequence of approximate minimizers of F_t converges to a minimizer of F_0 [38].

To show that $F_t \Gamma$ converges to F_0 , we need to show the following:

(1) Lower semicontinuity: for every sequence \mathbf{f}_t that converges to a configuration \mathbf{f} (in the weak $W^{2,2}$ topology),

$$\liminf_{t \rightarrow 0} F_t[\mathbf{f}_t] \geq F_0[\mathbf{f}].$$

(2) Recovery sequence: for every $\mathbf{f} \in W^{2,2}$, there exists a sequence \mathbf{f}_t that weakly converges to \mathbf{f} for which

$$\liminf_{t \rightarrow 0} F_t[\mathbf{f}_t] = F_0[\mathbf{f}].$$

To prove the lower-semicontinuity property, we note that the weak $W^{2,2}$ convergence of \mathbf{f}_t to \mathbf{f} implies the weak convergence of the corresponding metrics $g(\mathbf{f}_t) \rightarrow g(\mathbf{f})$, in the weak $W^{1,2}$ topology, which by the Sobolev embedding theorem [28] implies the convergence of the metrics in the strong C^0 topology, i.e., uniform convergence. It follows at once that the corresponding family of second fundamental forms weakly converges in L^2 to the second fundamental form of \mathbf{f} , $h(\mathbf{f}_t) \rightarrow h(\mathbf{f})$. Since the bending content is equivalent to an L^2 norm of the second fundamental form, it follows at once that

$$\liminf_{t \rightarrow 0} E_B[\mathbf{f}_t] \geq E_B[\mathbf{f}].$$

Now either $g(\mathbf{f}) = \bar{g}$, in which case

$$\lim_{t \rightarrow 0} F_t[\mathbf{f}_t] \geq \liminf_{t \rightarrow 0} E_B[\mathbf{f}_t] \geq E_B[\mathbf{f}] = F_0[\mathbf{f}],$$

or $g(\mathbf{f}) \neq \bar{g}$, in which case

$$\liminf_{t \rightarrow 0} F_t[\mathbf{f}_t] = \infty = F_0[\mathbf{f}].$$

To prove the existence of a recovery sequence we take, given \mathbf{f} , the constant sequence $\mathbf{f}_t = \mathbf{f}$. If $g(\mathbf{f}) = \bar{g}$ then

$$\lim_{t \rightarrow 0} F_t[\mathbf{f}_t] = E_B[\mathbf{f}] = F_0[\mathbf{f}].$$

If, however, $g(\mathbf{f}) \neq \bar{g}$ then

$$\lim_{t \rightarrow 0} F_t[\mathbf{f}_t] = \infty = F_0[\mathbf{f}].$$

Comments.

(1) The assumption that the reference metric can be embedded isometrically with finite bending is by no means trivial. The Nash-Kuiper embedding theorem only guarantees the existence of a C^1 embedding. Embeddings of class $W^{2,2}$ have been shown to exist under additional assumptions (see, e.g., [39]); however, there is no general existence proof for arbitrary metrics.

(2) We use the weak $W^{2,2}$ topology because we aim to eventually prove that *every* family of approximate minimizers has a converging subsequence (implying that the limit is a minimizer of the Willmore functional). Such a compactness result cannot possibly hold in the strong $W^{2,2}$ topology.

(3) A side result of the above theorem is that the Willmore functional has a minimizer (although not necessarily unique), even if the functionals F_t do not have minimizers.

APPENDIX B: PERTURBATION ANALYSIS

Consider a sufficiently regular surface $f(x^1, x^2)$. Any small enough perturbation can be decomposed into a sum of in-plane and out-of-plane displacements,

$$\delta f = v^\gamma \partial_\gamma f + w \mathcal{N}, \quad (\text{B1})$$

where \mathcal{N} is the unit vector normal to the surface. Given such a perturbation, we are going to calculate the variation in the elastic energy.

In order to retain the tensorial nature of the problem (coordinate invariance), we need to only utilize a covariant differentiation. To do so, we need to specify a metric with respect to which the Christoffel symbols are defined. It turns out that choosing the (natural) induced metric on the surface yields the most compact form for the variation in energy.

We recall the definitions of the covariant derivatives of a scalar field W , a contravariant vector V^γ , a covariant vectors V_γ , and a mixed tensor T^β_γ ,

$$\nabla_\alpha W = \partial_\alpha W,$$

$$\nabla_\alpha V^\gamma = \partial_\alpha V^\gamma + \Gamma_{\alpha\beta}^\gamma V^\beta,$$

$$\nabla_\alpha V_\gamma = \partial_\alpha V_\gamma - \Gamma_{\alpha\gamma}^\beta V_\beta,$$

$$\nabla_\alpha T^\beta_\gamma = \partial_\alpha T^\beta_\gamma + \Gamma_{\alpha\delta}^\beta T^\delta_\gamma - \Gamma_{\alpha\gamma}^\delta T^\beta_\delta. \quad (\text{B2})$$

Note that ∇_α and ∇_β commute only when operating on scalars. As operators on higher rank tensors, their commutator is nonzero and relates to the Gaussian curvature of the surface.

To calculate the variation in energy, we need to calculate the variation in the two fundamental forms. For this, we use the Gauss-Weingarten equations,

$$\partial_\alpha \partial_\beta f = \Gamma_{\alpha\beta}^\gamma \partial_\gamma f + h_{\alpha\beta} \mathcal{N}, \quad \partial_\alpha \mathcal{N} = -T^\beta_\alpha \partial_\beta f,$$

where $T^\beta_\alpha = (g^{-1})^{\beta\gamma} h_{\gamma\alpha}$. (Note that by our definitions of index raising, $T^\beta_\alpha = h^\beta_\alpha$ only if $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$.) It follows that for a vector in \mathbb{R}^3 in the form

$$\mathbf{v} = a^\alpha \partial_\alpha f + b \mathcal{N},$$

its derivative is given by

$$\partial_\beta \mathbf{v} = (\nabla_\beta a^\alpha - b T^\alpha_\beta) \partial_\alpha f + (\nabla_\beta b + a^\alpha h_{\alpha\beta}) \mathcal{N}. \quad (\text{B3})$$

1. Variation in stretching content

Differentiating Eq. (B1) and using Eq. (B3), we get

$$\partial_\alpha \delta f = [(\nabla_\alpha v^\gamma) - w T^\gamma_\alpha] \partial_\gamma f + [(\nabla_\alpha w) + v^\gamma h_{\alpha\gamma}] \mathcal{N}, \quad (\text{B4})$$

from which we derive the variation in the metric,

$$\begin{aligned} \delta g_{\alpha\beta} &= \partial_\alpha f \cdot \partial_\beta \delta f + \partial_\alpha \delta f \cdot \partial_\beta f + \partial_\alpha \delta f \cdot \partial_\beta \delta f \\ &= g_{\beta\gamma} (\nabla_\alpha v^\gamma) + g_{\alpha\gamma} (\nabla_\beta v^\gamma) - 2w h_{\alpha\beta} \\ &\quad + [(\nabla_\alpha w) + v^\gamma h_{\alpha\gamma}] [(\nabla_\beta w) + v^\delta h_{\beta\delta}] \\ &\quad + [(\nabla_\alpha v^\gamma) - w T^\gamma_\alpha] g_{\gamma\delta} [(\nabla_\beta v^\delta) - w T^\delta_\beta]. \end{aligned} \quad (\text{B5})$$

Substituting into Eq. (5), we obtain the variation in stretching content density,

$$\begin{aligned} \delta w_S &= \frac{1}{2} s^{\alpha\beta} \delta g_{\alpha\beta} + \frac{1}{8} \mathcal{A}^{\alpha\beta\gamma\delta} \delta g_{\alpha\beta} \delta g_{\gamma\delta} \\ &= \delta w_S^{(1,0)}(v) + \delta w_S^{(0,1)}(w) + \delta w_S^{(2,0)}(v) + \delta w_S^{(0,2)}(w) \\ &\quad + \delta w_S^{(1,1)}(v, w) + \delta w_S^{(1,2)}(v, w) + \delta w_S^{(0,3)}(w) + \delta w_S^{(0,4)}(w) \\ &\quad + \mathcal{O}(v^3, v^2 w, v w^3, w^5), \end{aligned} \quad (\text{B6})$$

where the various $\delta w_S^{(i,j)}$ represent terms of different orders in v and w ,

$$\delta w_S^{(1,0)}(v) = s^{\alpha\beta} g_{\beta\gamma} (\nabla_\alpha v^\gamma),$$

$$\delta w_S^{(0,1)}(w) = -s^{\alpha\beta} h_{\alpha\beta} w,$$

$$\begin{aligned} \delta w_S^{(2,0)}(v) &= \frac{1}{2} s^{\alpha\beta} h_{\alpha\gamma} h_{\beta\eta} v^\gamma v^\eta + \frac{1}{2} s^{\alpha\beta} g_{\gamma\epsilon} (\nabla_\alpha v^\gamma) (\nabla_\beta v^\epsilon) \\ &\quad + \frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} g_{\beta\eta} g_{\delta\epsilon} (\nabla_\alpha v^\eta) (\nabla_\gamma v^\epsilon), \end{aligned}$$

$$\begin{aligned} \delta w_S^{(0,2)}(w) &= \frac{1}{2} s^{\alpha\beta} [(\nabla_\alpha w) (\nabla_\beta w) + H_{\alpha\beta} w^2] \\ &\quad + \frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta} w^2, \end{aligned}$$

$$\begin{aligned} \delta w_S^{(1,1)}(v, w) &= s^{\alpha\beta} h_{\beta\gamma} v^\gamma (\nabla_\alpha w) - s^{\alpha\beta} h_{\alpha\eta} (\nabla_\beta v^\eta) w \\ &\quad - \mathcal{A}^{\alpha\beta\gamma\delta} h_{\gamma\delta} g_{\beta\eta} (\nabla_\alpha v^\eta) w, \end{aligned}$$

$$\begin{aligned} \delta w_S^{(1,2)}(v, w) &= \frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} g_{\beta\eta} (\nabla_\alpha v^\eta) [(\nabla_\gamma w) (\nabla_\delta w) + H_{\gamma\delta} w^2] \\ &\quad - \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\delta\mu} v^\mu w (\nabla_\gamma w) \\ &\quad + \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\delta\mu} w^2 (\nabla_\gamma v^\mu), \end{aligned}$$

$$\delta w_S^{(0,3)}(w) = -\frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} h_{\alpha\beta} [(\nabla_\gamma w) (\nabla_\delta w) + H_{\gamma\delta} w^2] w,$$

$$\begin{aligned} \delta w_S^{(0,4)}(w) &= \frac{1}{8} \mathcal{A}^{\alpha\beta\gamma\delta} [(\nabla_\alpha w) (\nabla_\beta w) + H_{\alpha\beta} w^2] \\ &\quad \times [(\nabla_\gamma w) (\nabla_\delta w) + H_{\gamma\delta} w^2], \end{aligned}$$

where we have used the symmetry of $s^{\alpha\beta}$ and $\mathcal{A}^{\alpha\beta\gamma\delta}$, and introduced the following new symmetric tensor,

$$H_{\alpha\beta} = T^\gamma_\alpha h_{\gamma\beta} = (g^{-1})^{\gamma\delta} h_{\delta\alpha} h_{\gamma\beta},$$

which is known as the third quadratic form.

Perturbation of a flat surface. When the unperturbed surface is flat then $h_{\alpha\beta} = H_{\alpha\beta} = 0$, which simplifies Eq. (B6) considerably. In particular, all the terms that are odd functions of the out-of-plane perturbation w vanish.

Perturbation of an isometric immersion. If the unperturbed surface is an isometric immersion $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$ then $s^{\alpha\beta} = 0$, which implies that the lowest-order nonvanishing terms in Eq. (B6) are

$$\begin{aligned} \delta w_S = & \frac{1}{2} \mathcal{A}^{\alpha\beta\gamma\delta} [g_{\beta\gamma} (\nabla_\alpha v^\gamma) - h_{\alpha\beta} w] [g_{\gamma\epsilon} (\nabla_\delta v^\epsilon) - h_{\gamma\delta} w] \\ & + \mathcal{O}(v^3, v^2 w, v w^2, w^3). \end{aligned}$$

Note that it is explicitly assumed here that derivatives of the deviations v, w are of the same order of magnitude as the deviation themselves. This assumption breaks down in the presence of small-scale features such as boundary layers.

2. Variation in bending content

To calculate the variation in the second quadratic form, we start with

$$h_{\alpha\beta} = \partial_\alpha \partial_\beta \mathbf{f} \cdot \mathcal{N} = -\partial_\beta \mathbf{f} \cdot \partial_\alpha \mathcal{N},$$

from which follows that

$$\delta h_{\alpha\beta} = -\partial_\beta \delta \mathbf{f} \cdot \partial_\alpha \mathcal{N} - \partial_\beta \mathbf{f} \cdot \partial_\alpha \delta \mathcal{N} - \partial_\beta \delta \mathbf{f} \cdot \partial_\alpha \delta \mathcal{N}. \quad (\text{B7})$$

The first term follows directly from Eq. (B4) and the Weingarten equation,

$$-\partial_\beta \delta \mathbf{f} \cdot \partial_\alpha \mathcal{N} = h_{\alpha\gamma} (\nabla_\beta v^\gamma) - w H_{\alpha\beta}. \quad (\text{B8})$$

To calculate the second and third terms, we need to express the perturbation $\delta \mathcal{N}$ in the unit normal vector. For that, we use the facts that $\delta(\mathcal{N} \cdot \mathcal{N}) = \delta(\partial_\alpha \mathbf{f} \cdot \mathcal{N}) = 0$; hence,

$$2\mathcal{N} \cdot \delta \mathcal{N} + \delta \mathcal{N} \cdot \delta \mathcal{N} = \partial_\alpha \delta \mathbf{f} \cdot \mathcal{N} + \partial_\alpha \mathbf{f} \cdot \delta \mathcal{N} + \partial_\alpha \delta \mathbf{f} \cdot \delta \mathcal{N} = 0.$$

Setting

$$\delta \mathcal{N} = (g^{-1})^{\gamma\delta} a_\delta \partial_\gamma \mathbf{f} + b \mathcal{N}, \quad (\text{B9})$$

this yields a closed set of equations for the three coefficients a_γ, b ,

$$\begin{aligned} a_\alpha = & -(1+b)[(\nabla_\alpha w) + v^\gamma h_{\alpha\gamma}] - a_\beta [(\nabla_\alpha v^\beta) - w T_\alpha^\beta], \\ b = & -\frac{1}{2} (g^{-1})^{\gamma\beta} a_\gamma a_\beta - \frac{1}{2} b^2. \end{aligned} \quad (\text{B10})$$

Applying Eqs. (B3) on Eq. (B9), we get

$$\partial_\alpha \delta \mathcal{N} = [(g^{-1})^{\gamma\delta} \nabla_\alpha a_\delta - b T_\alpha^\gamma] \partial_\gamma \mathbf{f} + [\nabla_\alpha b + T_\alpha^\delta a_\delta] \mathcal{N}, \quad (\text{B11})$$

where we have used the fact that the covariant derivative of $g_{\alpha\beta}$ and its inverse vanishes. It follows at once that the second term in Eq. (B7) is given by

$$-\partial_\beta \mathbf{f} \cdot \partial_\alpha \delta \mathcal{N} = -(\nabla_\alpha a_\beta) + b h_{\alpha\beta}. \quad (\text{B12})$$

The third term in Eq. (B7) is obtained by combining Eqs. (B11) and (B4),

$$\begin{aligned} -\partial_\beta \delta \mathbf{f} \cdot \partial_\alpha \delta \mathcal{N} = & -[(\nabla_\beta v^\gamma) - w T_\beta^\gamma] [(\nabla_\alpha a_\gamma) - b h_{\alpha\gamma}] \\ & - [(\nabla_\beta w) + v^\gamma h_{\beta\gamma}] [(\nabla_\alpha b) + T_\alpha^\delta a_\delta]. \end{aligned} \quad (\text{B13})$$

It remains to combine Eqs. (B8), (B12), and (B13) to get

$$\begin{aligned} \delta h_{\alpha\beta} = & h_{\alpha\gamma} (\nabla_\beta v^\gamma) - w H_{\alpha\beta} - (\nabla_\alpha a_\beta) + b h_{\alpha\beta} - [(\nabla_\beta v^\gamma) - w T_\beta^\gamma] \\ & \times [(\nabla_\alpha a_\gamma) - b h_{\alpha\gamma}] - [(\nabla_\beta w) + v^\gamma h_{\beta\gamma}] [(\nabla_\alpha b) + T_\alpha^\delta a_\delta]. \end{aligned} \quad (\text{B14})$$

So far all the relations are exact, i.e., no assumptions have been made about the smallness of the perturbation, other than the ability to decompose it in the form (B1). Equations (B10) are a set of three quadratic equations for a_γ, b , which we may solve by successive approximations.

Perturbation of a flat surface. When the surface is flat $h_{\alpha\beta} = 0$, Eqs. (B10) and (B14) reduce into

$$a_\alpha = -(\nabla_\alpha w) - b(\nabla_\alpha w) - a_\beta (\nabla_\alpha v^\beta),$$

$$b = -\frac{1}{2} (g^{-1})^{\gamma\beta} a_\gamma a_\beta - \frac{1}{2} b^2,$$

and

$$\delta h_{\alpha\beta} = -(\nabla_\alpha a_\beta) - (\nabla_\beta v^\gamma) (\nabla_\alpha a_\gamma) - (\nabla_\beta w) (\nabla_\alpha b).$$

Solving for a_α, b by successive approximation, we get

$$\begin{aligned} a_\alpha = & -(\nabla_\alpha w) + (\nabla_\beta w) (\nabla_\alpha v^\beta) + \frac{1}{2} (g^{-1})^{\beta\gamma} (\nabla_\alpha w) (\nabla_\beta w) (\nabla_\gamma w) \\ & + \mathcal{O}(v^2, v w^2, w^4), \end{aligned}$$

$$b = -\frac{1}{2} (g^{-1})^{\gamma\beta} (\nabla_\beta w) (\nabla_\gamma w) + \mathcal{O}(v^2, v w, w^3).$$

Putting it all together, we obtain a simple expression for the variation in the second form,

$$\begin{aligned} \delta h_{\alpha\beta} = & (\nabla_\alpha \nabla_\beta w) - (\nabla_\gamma w) (\nabla_\alpha \nabla_\beta v^\gamma) \\ & - \frac{1}{2} (g^{-1})^{\gamma\delta} (\nabla_\alpha \nabla_\beta w) (\nabla_\gamma w) (\nabla_\delta w) + \mathcal{O}(v^2, v w^2, w^4). \end{aligned} \quad (\text{B15})$$

We then substitute Eqs. (B15) into Eq. (5) to calculate the variation in the bending content density,

$$\begin{aligned} \delta w_B = & \delta w_B^{(0,2)}(w) + \delta w_B^{(1,2)}(v, w) + \delta w_B^{(0,4)}(w) \\ & + \mathcal{O}(v^3, v^2 w, v w^3, w^5), \end{aligned} \quad (\text{B16})$$

where

$$\delta w_B^{(0,2)}(w) = \frac{1}{24} \mathcal{A}^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha w) (\nabla_\delta \nabla_\gamma w),$$

$$\delta w_B^{(1,2)}(v, w) = -\frac{1}{12} \mathcal{A}^{\alpha\beta\gamma\delta} (\nabla_\beta \nabla_\alpha v^\eta) (\nabla_\eta w) (\nabla_\gamma \nabla_\delta w),$$

$$\begin{aligned} \delta w_B^{(0,4)}(w) = & -\frac{1}{24} \mathcal{A}^{\alpha\beta\gamma\delta} (g^{-1})^{\eta\epsilon} (\nabla_\eta w) (\nabla_\epsilon w) (\nabla_\beta \nabla_\alpha w) \\ & \times (\nabla_\delta \nabla_\gamma w). \end{aligned}$$

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